McGill

June 2020
Final Examination

Models for Financial Economics
Economics 765

Take-home exam due on or before Thursday June 25th 2020.

This exam comprises 5 pages, including the cover page

1. Use the probabilistic argument in Shreve section 5.2.5 to obtain an explicit expression for the function $p(t, x)$, which gives the value of a European put option at time t when the price $S(t)$ of the underlying asset is x. The option expires at time T with strike price K. (Do not rely on put-call parity to obtain the answer.)

Similarly, use a probabilistic argument to obtain the explicit form of the function $f(t, x)$ that gives the value of a forward contract that expires at time T with strike price K.

2. The binomial asset pricing model supposes that, in going from period t to period $t + 1$, the stock price may move from S_t to S_{t+1} where $S_{t+1} = uS_t$ or $S_{t+1} = dS_t$, with $d < 1+r < u$, where r is the risk-free interest rate. Construct the risk-neutral measure for this model using only the requirement that the discounted stock price must be a martingale under it.

Consider a trinomial asset pricing model, in which, starting from S_t there are three distinct possibilities: S_{t+1} can be uS_t, mS_t, or dS_t, with $d < m < u$ and $d < 1 + r < u$.

(i) Does there exist a risk-neutral measure for this model of the stock price, and, if so, is it unique?

(ii) If there is no risk-neutral measure, discover an arbitrage. If there is, and it is not unique, give an example of a derivative security, the random value of which is a deterministic function of the stock price at maturity, that cannot be hedged by a portfolio containing only the stock and a sum of money in the risk-free account with interest rate r. If there is a unique risk-neutral measure, say explicitly how the hedging portfolio is constructed for a European call option.

(iii) In the context of the trinomial model, in which in period $t + 1$ there are three possibilities for every one possibility in period t, suppose now that there are two different stocks, each with its own parameters d_i, m_i, u_i, $i = 1, 2$. How are your answers to (i) and (ii) changed in this circumstance?

3. This exercise is taken from Shreve, exercise 5.10. Let the price at time t of a European call expiring at time T with strike price K be denoted by $C(t)$. Similarly, denote by $P(t)$ the price at time t of a European put expiring at T with strike price K. The put-call parity relationship says that $C(t) - P(t) = F(t)$, where $F(t)$ is the price at t of a forward contract for delivery of one share of stock at time T at price K. Thus

$$F(t) = \widetilde{\mathrm{E}}\big[\mathrm{e}^{-r(T-t)}\big(S(T) - K\big) \;\big|\; \mathcal{F}(t)\big].$$

in standard notation. Because $\mathrm{e}^{-rt}S(t)$ is a martingale under the risk-neutral measure, we see that $F(t) = S(t) - \mathrm{e}^{-r(T-t)}K$.

Now consider a fixed date t_0 between 0 and T, and consider a *chooser option*, which gives its holder the right at time t_0 to choose to own either the call or the put expiring at T with strike price K.

(i) Show that at time t_0 the value of the chooser option is

$$C(t_0) + \max\bigl[0, -F(t_0)\bigr] = C(t_0) + \bigl(\mathrm{e}^{-r(T-t_0)}K - S(t_0)\bigr)_+.$$

(ii) Show that the value of the chooser option at time 0 is the sum of the value of a call expiring at time T with strike price K and the value of a put expiring at time t_0 with strike price $\mathrm{e}^{-r(T-t_0)}K$.

4. Let $(\Omega, \mathcal{F}, P)$ be a probability space with Ω the unit interval $[0,1]$, $\mathcal{F}$ the Borel σ-algebra $\mathcal{B}$, and P Lebesgue measure. Let $\mathcal{F}_n$ be the σ-algebra generated by the intervals of the form $\bigl[(j-1)2^{-n}, j2^{-n}\bigr]$, $j = 1, 2, \ldots, 2^n$, open to the left, closed to the right. Let X be a bounded continuous function on $[0,1]$.

(i) Give the explicit form of the conditional expectation $\mathrm{E}(X \,|\, \mathcal{F}_n)$.

(ii) Show that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all n.

(iii) Show that the sequence of random variables $\{\mathrm{E}(X \,|\, \mathcal{F}_n)\}$, $n = 1, 2, \ldots$, converges for almost all $\omega \in [0,1]$. Characterise the limiting variable. **Hint:** Express ω as the infinite sum

$$\omega = \sum_{j=1}^{\infty} \omega_j 2^{-j}$$

where $\omega_j = 0$ or 1 for all j. Each sequence $\{\omega_j\}$ defines a unique real number ω, and each real $\omega \in [0,1]$ defines a unique sequence, provided that one excludes sequences such that there exists $J > 0$ finite with $\omega_J = 1$ and $\omega_j = 0$ for all $j > J$.

(iv) Show directly that $\mathrm{E}\bigl(\mathrm{E}(X \,|\, \mathcal{F}_n)\bigr) = \mathrm{E}(X)$.

5. The value function $v_L(x)$ for the perpetual American put was shown to be

$$v_L(x) = \begin{cases} K - x & 0 \le x \le L, \\ (K-L)(x/L)^{-2r/\sigma^2} & x \ge L. \end{cases} \tag{1}$$

Here x is the current stock price, K the strike price, and r and σ are the parameters of the geometric Brownian motion

$$\mathrm{d}S(t) = rS(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}\widetilde{W}(t) \tag{2}$$

that the stock price $S(t)$ is assumed to follow. As usual $\widetilde{W}(t)$ is a Brownian motion under the risk-neutral measure.

While the first case in (1) is in effect, the left-hand derivative of $v_L(x)$ at $x = L$ is $v_L'(L-) = -1$. What is the right-hand derivative at $x = L$, that is, the derivative from the second case? Show that the "smooth-pasting" condition

$$v_{L_*}'(L_*-) = v_{L_*}'(L_*+)$$

is satisfied only for $L_* = 2rK/(2r+\sigma^2)$.

Consider two perpetual American puts on the geometric Brownian motion (2). Suppose that the puts have different strike prices, K_1 and K_2, where $0 < K_1 < K_2$. Let $v_1(x)$ and $v_2(x)$ denote their respective prices. Show that $v_2(x)$ satisfies the first two linear complementarity conditions,

$$v_2(x) \geq (K_1 - x)_+ \text{ for all } x \geq 0, \tag{3}$$

$$rv_2(x) - rxv_2'(x) - \frac{1}{2}\sigma^2 x^2 v_2''(x) \geq 0 \text{ for all } x \geq 0, \tag{4}$$

for the perpetual American put price with strike K_1, but that $v_2(x)$ does not satisfy the third linear complementarity condition:

for each $x \geq 0$, equality holds in either (3) or (4) or both.

6. Let X be a real-valued random variable defined on the probability space $(\Omega, \mathcal{F}, P)$.

(i) Suppose that $\mathrm{E}|X|^p$ exists, where $p > 0$. Prove the *Markov inequality*, which states that, for all $\varepsilon > 0$,

$$P(|X| > \varepsilon) \leq \frac{\mathrm{E}|X|^p}{\varepsilon^p}. \tag{5}$$

(ii) Let $g : \mathbb{R} \to \mathbb{R}_+$ be an increasing function. Show that, for all real a,

$$P(X > a) \leq \frac{\mathrm{E}(g(X))}{g(a)}. \tag{6}$$

(iii) Suppose that a discrete-time filtration is defined on the measure space $(\Omega, \mathcal{F})$ as a nested set of σ-algebras $\mathcal{F}_t$, $t = 0, 1, 2, \ldots$, with $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$. A martingale in this discrete-time context is a stochastic process X_t, defined for $t = 0, 1, \ldots$, and such that X_t is $\mathcal{F}_t$–measurable for all t, and

$$\mathrm{E}\big(X_{t+1} \,|\, \mathcal{F}_t\big) = X_t. \tag{7}$$

Show that (7) implies that, for all $t = 0, 1, \ldots$,

$$\mathrm{E}\big(X_{t+s} \,|\, \mathcal{F}_t\big) = X_t, \quad s = 1, 2, \ldots. \tag{8}$$

(iv) A stopping time τ relative to the filtration $\mathcal{F}_t$ is a random variable that takes on the possible values $0, 1, 2, \ldots, \infty$. It is such that the event $\{\omega : \tau(\omega) = k\}$ belongs to $\mathcal{F}_k$ for all $k = 0, 1, \ldots$. The stopped process $X_{t\wedge\tau}$ is defined by the equation

$$X_{t\wedge\tau}(\omega) = X_{\min(t,\tau(\omega))}(\omega).$$

Show that

$$X_{t\wedge\tau} = \sum_{s=1}^{t-1} \mathrm{I}(\tau = s)X_s + \mathrm{I}(\tau \geq t)X_t.$$

If X_t is a martingale, show that the process $X_{t\wedge\tau}$ is also a martingale.

(v) The σ-algebra $\mathcal{F}_\tau$ defined by a stopping time τ is such that $A \in \mathcal{F}_\tau$ iff the event $A \cap \{\tau = t\} \in \mathcal{F}_t$ for all $t = 0, 1, 2, \ldots$. If $\tau < n$ almost surely for some finite positive integer n, show that, for all $t = 0, 1, \ldots, n$,

$$\mathrm{E}(X_t \,|\, \mathcal{F}_\tau) = \sum_{s=0}^{n} \mathrm{E}\big(X_t \mathrm{I}(\tau = s) \,|\, \mathcal{F}_s\big).$$

7. (Shreve exercises 11.1, 11.2, and more) Let $N(t)$ be a Poisson process with intensity λ and let $M(t)$ be the compensated Poisson process with $M(t) = N(t) - \lambda t$.

(i) Show that $M^2(t) - \lambda t$ is a martingale.

(ii) Show that $M^2(t)$ is a submartingale.

Suppose we have observed a Poisson process with intensity λ up to time s, have seen that $N(s) = k$, and are interested in the value of $N(s+t)$ for small positive t. Show that

$$P\{N(s+t) = k \,|\, N(s) = k\} = 1 - \lambda t + O(t^2),$$
$$P\{N(s+t) = k+1 \,|\, N(s) = k\} = \lambda t + O(t^2),$$
$$P\{N(s+t) \geq k+2 \,|\, N(s) = k\} = O(t^2),$$

where $O(t^2)$ is used to denote terms involving t^2 and higher powers of t.

Let N_1, N_2, …, N_m be a set of m mutually independent Poisson processes, with intensities λ_i, $i = 1, \ldots, m$, defined on the probability space $(\Omega, \mathcal{F}, P)$, and relative to the same filtration $\mathcal{F}(t)$, $t \geq 0$. Show that the process N defined by

$$N(t) = \sum_{i=1}^{m} N_i(t)$$

is a Poisson process with intensity $\lambda = \sum_{i=1}^{m} \lambda_i$.