

Economics 765

June 9, 2020

R. Davidson

Assignment 5

You are asked to do exercises 6.1, 6.6, 6.8, and 6.9 of Volume 2 of Shreve. The essence of these exercises is reproduced below for convenience.

6.1 Consider the stochastic differential equation

$$dX(u) = (a(u) + b(u)X(u)) du + (\gamma(u) + \sigma(u)X(u)) dW(u), \quad (6.2.4)$$

where $W(u)$ is a Brownian motion relative to a filtration $\mathcal{F}(u)$, $u \geq 0$, and we allow $a(u)$, $b(u)$, $\gamma(u)$, and $\sigma(u)$ to be processes adapted to this filtration. Fix an initial time t and an initial position $x \in \mathbb{R}$. Define

$$Z(u) = \exp \left\{ \int_t^u \sigma(v) dW(v) + \int_t^u \left(b(v) - \frac{1}{2} \sigma^2(v) \right) dv \right\},$$
$$Y(u) = x + \int_t^u \frac{a(v) - \sigma(v)\gamma(v)}{Z(v)} dv + \int_t^u \frac{\gamma(v)}{Z(v)} dW(v).$$

(i) Show that $Z(t) = 1$ and

$$dZ(u) = b(u)Z(u) du + \sigma(u)Z(u) dW(u), \quad u \geq t.$$

(ii) By its very definition, $Y(u)$ satisfies $Y(t) = x$ and

$$dY(u) = \frac{a(u) - \sigma(u)\gamma(u)}{Z(u)} du + \frac{\gamma(u)}{Z(u)} dW(u), \quad u \geq t.$$

Show that $X(u) = Y(u)Z(u)$ solves the stochastic differential equation (6.2.4) and satisfies the initial condition $X(t) = x$.

6.6 Problem on the moment-generating function for the Cox-Ingersoll-Ross process.

(i) Let W_1, \dots, W_d be independent Brownian motions and let a and σ be positive constants. For $j = 1, \dots, d$, let $X_j(t)$ be the solution of the *Ornstein-Uhlenbeck* stochastic differential equation

$$dX_j(t) = -\frac{b}{2}X_j(t) dt + \frac{1}{2}\sigma dW_j(t).$$

Show that

$$X_j(t) = e^{-\frac{1}{2}bt} \left[X_j(0) + \frac{\sigma}{2} \int_0^t e^{\frac{1}{2}bu} dW_j(u) \right].$$

Show further that, for fixed t , the random variable $X_j(t)$ is normal with

$$EX_j(t) = e^{-\frac{1}{2}bt} X_j(0), \quad \text{Var}(X_j(t)) = \frac{\sigma^2}{4b} [1 - e^{-bt}].$$

(Hint: Use Theorem 4.4.9.)

(ii) Define

$$R(t) = \sum_{j=1}^d X_j^2(t), \quad (6.9.18)$$

and show that

$$dR(t) = (a - bR(t)) dt + \sigma\sqrt{R(t)} dB(t),$$

where $a = d\sigma^2/4$ and

$$B(t) = \sum_{j=1}^d \int_0^t \frac{X_j(s)}{\sqrt{R(s)}} dW_j(s)$$

is a Brownian motion. In other words, $R(t)$ is a Cox-Ingersoll-Ross interest rate process. (Hint: Lévy's theorem to show that $B(t)$ is a Brownian motion.)

(iii) Suppose $R(0) > 0$ given, and define

$$X_j(0) = \sqrt{\frac{R(0)}{d}}.$$

Show then that $X_1(t), \dots, X_d(t)$ are independent, identically distributed, normal random variables, each having expectation

$$\mu(t) = e^{-\frac{1}{2}bt} \sqrt{\frac{R(0)}{d}}$$

and variance

$$v(t) = \frac{\sigma^2}{4b} [1 - e^{-bt}].$$

(iv) Show that the moment-generating function of the square of a variable X with the $N(\mu, \sigma^2)$ distribution is

$$\mathbb{E} \exp uX^2 = \frac{1}{\sqrt{1 - 2u\sigma^2}} \exp \left\{ \frac{u\mu^2}{1 - 2u\sigma^2} \right\}$$

for $u < 1/2\sigma^2$.

(v) Part (iii) shows that $R(t)$ given by (6.9.18) is the sum of squares of IID normal random variables and hence has a *noncentral χ^2 distribution*, the term “noncentral” referring to the fact that $\mu(t) = \mathbb{E}X_j(t)$ is not zero. Show that the moment-generating function of $R(t)$ is

$$\mathbb{E} \exp uR(t) = \left(\frac{1}{1 - 2v(t)u} \right)^{2a/\sigma^2} \exp \left\{ \frac{e^{-bt}uR(0)}{1 - 2v(t)u} \right\}$$

for $u < 1/(2v(t))$.

6.8 Consider the SDE

$$dX(u) = \beta(u, X(u)) du + \gamma(u, X(u)) dW(u).$$

We assume that if we begin a process at an arbitrary initial positive value $X(t) = x$ at an arbitrary initial time t , and let it evolve according to the SDE, its value at each time $T > t$ could be any positive number but cannot be nonpositive. For $0 \leq t < T$, let $p(t, T, x, y)$ be the transition density for the solution to this equation, by which we mean that the random variable $X(T)$ that evolves from $X(t) = x$ has density $p(t, T, x, y)$ in the y variable. We assume that $p(t, T, x, y) = 0$ for $0 \leq t < T$ and $y \leq 0$.

Show that $p(t, T, x, y)$ satisfies the *Kolmogorov backward equation*

$$-p_t(t, T, x, y) = \beta(t, x)p_x(t, T, x, y) + \frac{1}{2}\gamma^2(t, x)p_{xx}(t, T, x, y).$$

6.9 With the same setup as for the previous exercise, show that $p(t, T, x, y)$ satisfies the *Kolmogorov forward equation*, which we may write as

$$\frac{\partial}{\partial T}p(t, T, x, y) = -\frac{\partial}{\partial y}(\beta(T, y)p(t, T, x, y)) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(\gamma^2(T, y)p(t, T, x, y)).$$

In contrast to the Kolmogorov backward equation, in which T and y are held constant and the variables are t and x , here t and x are held constant and the variables are T and y .

Notes on this exercise:

- There is a misprint in Shreve in the forward equation: He writes $\beta(t, y)$ for $\beta(T, y)$.
- In the book, there is a long series of steps that can guide you to the desired result. You are at liberty to make use of this or not as you see fit.
- The partial differential operator in the forward equation is the *adjoint* of that in the backward equation. The latter is of course simpler to use.
- The forward equation is also referred to as the *Fokker-Planck equation*, after two eminent physicists.