

Economics 765

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Assignment 4

You are asked to do exercises 4.19, 5.5, 5.8, and 5.11 of Volume 2 of Shreve. The essence of these exercises is reproduced below for convenience.

4.19 Let $W(t)$ be a Brownian motion, and define

$$B(t) = \int_0^t \text{sign}(W(s)) dW(s),$$

where

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

- (i) Show that $B(t)$ is a Brownian motion.
- (ii) Use Itô's product rule to compute $d[B(t)W(t)]$. Integrate both sides of the resulting equation and take expectations. Show that $E[B(t)W(t)] = 0$ (so that $B(t)$ and $W(t)$ are uncorrelated).
- (iii) Verify that

$$dW^2(t) = 2W(t) dW(t) + dt.$$

- (iv) Use Itô's product rule to compute $d[B(t)W^2(t)]$. Integrate both sides of the resulting equation and take expectations to conclude that

$$E[B(t)W^2(t)] \neq EB(t) \cdot EW^2(t).$$

Explain why this shows that, although they are uncorrelated normal random variables, $B(t)$ and $W(t)$ are not independent.

5.5 You are asked to prove the following result, Corollary 5.3.2 of Shreve.

Let $W(t)$, $0 \leq t \leq T$, be a Brownian motion on a probability space (Ω, \mathcal{F}, P) , and let $\mathcal{F}(t)$, $0 \leq t \leq T$, be the filtration generated by this Brownian motion. Let $\Theta(t)$, $0 \leq t \leq T$, be an adapted process, define

$$Z(t) = \exp \left\{ - \int_0^t \Theta(u) dW(u) - \frac{1}{2} \int_0^t \Theta^2(u) du \right\},$$
$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u) du,$$

and assume that $\tilde{\mathbb{E}} \int_0^T \Theta^2(u) Z^2(u) du < \infty$. Set $Z = Z(T)$. Then $\mathbb{E}Z = 1$, and under the probability measure \tilde{P} defined by

$$\tilde{P}(A) = \int_A Z(\omega) dP(\omega) \quad \text{for all } A \in \mathcal{F},$$

the process $\tilde{W}(t)$, $0 \leq t \leq T$, is a Brownian motion.

Now let $\tilde{M}(t)$, $0 \leq t \leq T$, be a martingale under \tilde{P} . Then there is an adapted process $\tilde{\Gamma}(u)$, $0 \leq u \leq T$, such that

$$\tilde{M}(t) = \tilde{M}(0) + \int_0^t \tilde{\Gamma}(u) d\tilde{W}(u), \quad 0 \leq t \leq T. \quad (1)$$

The suggested steps for the proof are as follows.

- (i) Compute the differential of $1/Z(t)$.
- (ii) Let $\tilde{M}(t)$, $0 \leq t \leq T$, be a martingale under \tilde{P} . Show that $M(t) = Z(t)\tilde{M}(t)$ is a martingale under P .
- (iii) According to Shreve's Theorem 5.3.1 (the one-dimensional martingale representation theorem), there is an adapted process $\Gamma(u)$, $0 \leq u \leq T$, such that

$$M(t) = M(0) + \int_0^t \Gamma(u) dW(u), \quad 0 \leq t \leq T.$$

Write $\tilde{M}(t) = M(t)(1/Z(t))$ and take its differential using Itô's product rule.

- (iv) Show that the differential of $\tilde{M}(t)$ is the sum of an adapted process, which we call $\tilde{\Gamma}(t)$, times $d\tilde{W}(t)$, and zero times dt . Integrate to obtain (1).

5.8 (Usual setup and notation.) Assume that there is a unique risk-neutral measure \tilde{P} , and let $\tilde{W}(t)$, $0 \leq t \leq T$, be the Brownian motion under \tilde{P} obtained by using Girsanov's theorem.

Now let $V(T)$ be an almost surely positive $\mathcal{F}(T)$ -measurable random variable (under both of the equivalent measures P and \tilde{P}). According to the risk-neutral pricing formula, the price at time t of a security paying $V(T)$ at time T is

$$V(t) = \tilde{\mathbb{E}} \left[V(T) \exp - \int_t^T R(u) du \mid \mathcal{F}(t) \right], \quad 0 \leq t \leq T.$$

- (i) Show that there exists an adapted process $\tilde{\Gamma}(t)$, $0 \leq t \leq T$, such that

$$dV(t) = R(t)V(t) dt + \frac{\tilde{\Gamma}(t)}{D(t)} d\tilde{W}(t), \quad 0 \leq t \leq T.$$

- (ii) Show that, for each $t \in [0, T]$, the price of the derivative security $V(t)$ at time t is almost surely positive.
- (iii) Conclude from (i) and (ii) that there exists an adapted process $\sigma(t)$, $0 \leq t \leq T$, such that

$$dV(t) = R(t)V(t) dt + \sigma(t)V(t) d\widetilde{W}(t), \quad 0 \leq t \leq T.$$

In other words, prior to time T , the price of every asset with almost surely positive price at time T follows a generalised geometric Brownian motion.

5.11 Consider a stock price process whose differential is

$$dS(t) = \alpha(t)S(t) dt + \sigma(t)S(t) dW(t), \quad 0 \leq t \leq T.$$

Suppose that an agent must pay a cash flow at rate $C(t)$ at each time t , where $C(t)$, $0 \leq t \leq T$, is an adapted process (as are $\alpha(t)$, the interest rate process $R(t)$, and $\sigma(t)$). If the agent holds $\Delta(t)$ shares of stock at each time t , then the differential of the portfolio value is

$$dX(t) = \Delta(t) dS(t) + R(t)(X(t) - \Delta(t)S(t)) dt - C(t) dt.$$

Show that there is a nonrandom value of $X(0)$ and a portfolio process $\Delta(t)$, $0 \leq t \leq T$, such that $X(T) = 0$ almost surely. (Hint: Define the risk-neutral measure and apply the result of exercise 5.5 to the process

$$\widetilde{M}(t) = \widetilde{\mathbb{E}} \left[\int_0^T D(u)C(u) du \mid \mathcal{F}(t) \right], \quad 0 \leq t \leq T,$$

where $D(t)$ is the discount process $\exp - \int_0^t R(s) ds$.)