Economics 765

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Assignment 2

You are asked to do exercises 2.5, 2.7, 2.8, and 2.11 of Volume 2 of Shreve. The essence of these exercises is reproduced below for convenience.

2.5 Let (X, Y) be a pair of random variables with joint density function

$$f_{X,Y}(x, y) = \begin{cases} \frac{2|x|+y}{\sqrt{2\pi}} \exp\left(-\frac{(2|x|+y)^2}{2}\right) & \text{if } y \geq -|x|, \\ 0 & \text{if } y < -|x|. \end{cases}$$

Show that X and Y are standard normal variables and that they are uncorrelated but not independent.

2.7 Let Y be an integrable random variable on a probability space $(\Omega, \mathcal{F}, P)$, and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Based on the information in $\mathcal{G}$, we can form the estimate $\mathrm{E}(Y|\mathcal{G})$ of Y and define the error of the estimation $\mathrm{Err} = Y - \mathrm{E}(Y|\mathcal{G})$. This is a random variable with expectation zero and some variance $\mathrm{Var(Err)}$. Let X be some other $\mathcal{G}$–measurable random variable, which we can regard as another estimate of Y. Show that

$$\mathrm{Var(Err)} \leq \mathrm{Var}(Y - X).$$

In other words, the estimate $\mathrm{E}(Y|\mathcal{G})$ minimises the variance of the error among all estimates based on the information in $\mathcal{G}$. (Hint: Let $\mu = \mathrm{E}(Y - X)$. Compute the variance of $Y - X$ as

$$\mathrm{E}\big[(Y - X - \mu)^2\big] = \mathrm{E}\Big[\big((Y - \mathrm{E}(Y|\mathcal{G})) + (\mathrm{E}(Y|\mathcal{G}) - X - \mu)\big)^2\Big].$$

Multiply out the right-hand side and use iterated conditioning to show that the cross term is zero.)

2.8 Let X and Y be integrable random variables on a probability space $(\Omega, \mathcal{F}, P)$. Then $Y = Y_1 + Y_2$, where $Y_1 = \mathrm{E}(Y|X)$ is $\sigma(X)$–measurable and $Y_2 = Y - \mathrm{E}(Y|X)$. Show that Y_2 and X are uncorrelated. More generally, show that Y_2 is uncorrelated with every $\sigma(X)$–measurable random variable.

2.11 (i) Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$, and let W be a nonnegative $\sigma(X)$–measurable random variable. Show that there exists a function g such that $W = g(X)$. (Hint: Recall that every set in $\sigma(X)$ is of the form $\{X \in B\}$ for some Borel set $B \in \mathbb{R}$. Suppose first that W is the indicator of such a set, and then use the standard machine.)

(ii) Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$, and let Y be a nonnegative random variable on this space. We do not assume that X and Y have a joint density. Nonetheless, show that there is a function g such that $\mathrm{E}(Y|X) = g(X)$.