Economics 765

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Assignment 1

You are asked to do exercise 1.3 from Volume 1 of Shreve, and exercises 1.1, 1.5, 1.9, 1.10, and 1.14 of Volume 2. The essence of these exercises is reproduced below for convenience.

1.3 In the one-period binomial model (as considered in the first class), suppose we want to determine the price at time zero of the derivative security with payoff $V_1 = S_1$. This means that the derivative security pays the stock price, in either state of the world. It can also be interpreted as a European call option with strike price $K = 0$. (Why? Be sure you understand this.) Use the risk-neutral pricing formula

$$V_0 = \frac{1}{1+r}\big[\tilde{p}V_1(H) + \tilde{q}V_1(T)\big]$$

to compute the time-zero price V_0 of this option.

Characterise the hedging portfolio that replicates the payoff of this derivative security.

1.1 The axioms for a probability measure P defined on a measure space $(\Omega, \mathcal{F})$ can be stated as follows:

(i) $P(\Omega) = 1$, and

(ii) whenever $A_1, A_2, \ldots$ is a sequence of disjoint sets in $\mathcal{F}$, then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n).$$

Use these axioms to show the following.

(i) If $A \in \mathcal{F}$, $B \in \mathcal{F}$, and $A \subset B$, then $P(A) \leq P(B)$.

(ii) If $A \in \mathcal{F}$ and $\{A_n\}_{n=1}^{\infty}$ is a sequence of sets in $\mathcal{F}$ with $\lim_{n\to\infty} P(A_n) = 0$ and $A \subset A_n$ for every n, then $P(A) = 0$.

1.5 When dealing with double Lebesgue integrals, just as with double Riemann integrals, the order of integration can be reversed. The only assumption required is that the function being integrated should be either nonnegative or integrable. Here is an application of this fact.

Let X be a nonnegative random variable with cumulative distribution function $F(x) = P(X \leq x)$. Show that

$$\mathrm{E}X = \int_0^{\infty} (1 - F(x))\,\mathrm{d}x$$

by showing that

$$\int_{\Omega}\int_0^{\infty} \mathrm{I}_{[0,X(\omega)[}(x)\,\mathrm{d}x\,\mathrm{d}P(\omega)$$

is equal to both $\mathrm{E}X$ and $\int_0^{\infty}(1 - F(x))\,\mathrm{d}x$.

1.9 Suppose that X is a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$, that $A \in \mathcal{F}$, and that, for every Borel subset B of $\mathbb{R}$, we have

$$\int_A \mathrm{I}_B(X(\omega))\,\mathrm{d}P(\omega) = P(A) \cdot P\{X \in B\}.$$

Then we say that X is *independent* of the event A.

Show that, if X is independent of an event A, then

$$\int_A g(X(\omega))\,\mathrm{d}P(\omega) = P(A) \cdot \mathrm{E}g(X)$$

for every nonnegative, Borel-measurable, function g.

1.10 Let P be the uniform Lebesgue measure on $\Omega = [0, 1]$. Define

$$Z(\omega) = \begin{cases} 0 & \text{if } 0 \le \omega < \frac{1}{2} \\ 2 & \text{if } \frac{1}{2} \le \omega \le 1. \end{cases}$$

For $A \in \mathcal{B}[0, 1]$, define

$$\tilde{P}(A) = \int_A Z(\omega)\,\mathrm{d}P(\omega).$$

(i) Show that $\tilde{P}$ is a probability measure.

(ii) Show that, if $P(A) = 0$, then $\tilde{P}(A) = 0$. We say that $\tilde{P}$ is *absolutely continuous* with respect to P.

(iii) Show that there is a set A for which $\tilde{P}(A) = 0$ but $P(A) > 0$. In other words, P and $\tilde{P}$ are not equivalent.

1.14 Let X be a nonnegative random variable defined on a probability space $(\Omega, \mathcal{F}, P)$ with the *exponential distribution*, which is

$$P\{X \le a\} = 1 - \mathrm{e}^{-\lambda a}, \quad a \ge 0.$$

where λ is a positive constant. Let $\tilde{\lambda}$ be another positive constant, and define

$$Z = \frac{\tilde{\lambda}}{\lambda}\,\mathrm{e}^{-(\tilde{\lambda} - \lambda)X}.$$

Define $\tilde{P}$ by

$$\tilde{P}(A) = \int_A Z\,\mathrm{d}P \quad \text{for all } A \in \mathcal{F}.$$

(i) Show that $\tilde{P}(\Omega) = 1$.

(ii) Compute the distribution function

$$\tilde{P}\{X \le a\} \text{ for } a \ge 0$$

for the random variable X under the probability measure $\tilde{P}$.