Economics 662D1

October 22, 2020 **R. Davidson**

Midterm Examination

Your completed exam should be uploaded to myCourses by 13.00 on October 22. Please submit two files per student: one, which should be a PDF file, with your written answers, and another, which may or may not be a simple text file, with your computer code. These files must be all your own work. You may make use of whatever non-human resources you wish, but you must not ask for or receive any help from any other person.

All students in this course have the right to submit in English or in French any written work that is to be graded.

Tou(te)s les étudiant(e)s qui suivent ce cours on le droit de soumettre tout travail écrit en français ou en anglais.

Academic Integrity statement [approved by Senate on 29 January 2003]:

McGill University values academic integrity. Therefore all students must understand the meaning and consequences of cheating, plagiarism and other academic offences under the Code of Student Conduct and Disciplinary Procedures.

L'université McGill attache une haute importance à l'honnêteté académique. Il incombe par conséquent à tou(te)s les étudiant(e)s de comprendre ce que l'on entend par tricherie, plagiat et autres infractions académiques, ainsi que les conséquences que peuvent avoir de telles actions, selon le Code de conduite de l'étudiant(e) et des procédures disciplinaires.

Answer all five questions in this exam.

Faites tous les cinq exercices de cet examen.

1. This question concerns estimators of the parameters of the linear regression model

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{u}, \quad \boldsymbol{u} \sim \mathrm{IID}(\boldsymbol{0}, \sigma^2\mathbf{I}).$$

The estimators considered by the Gauss-Markov Theorem are linear and unbiased, so that they can be written as $\hat{\boldsymbol{\beta}} = \boldsymbol{Ay}$, with $\boldsymbol{AX} = \mathbf{I}$. Show that this class of estimators is in fact identical to the class of estimators of the form

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{W}^\top\boldsymbol{X})^{-1}\boldsymbol{W}^\top\boldsymbol{y},$$

where $\boldsymbol{W}$ is a matrix of exogenous variables such that $\boldsymbol{W}^\top\boldsymbol{X}$ is nonsingular.

2. The file located at the URL

https://russell-davidson.arts.mcgill.ca/e662/e662.midterm.20.dat

contains 113 observations on two variables, C_t, $t = 1, \ldots, 113$, consumption expenditure, and Y_t, disposable income, in that order. Construct the variables c_t, the logarithm of consumption, and y_t, the logarithm of income. Use them to estimate the following **autoregressive distributed lag** model:

$$c_t = \alpha + \beta c_{t-1} + \gamma_0 y_t + \gamma_1 y_{t-1} + u_t, \quad t = 2, \ldots, 113. \tag{1}$$

Such models are often expressed in first-difference form, that is, as

$$\Delta c_t = \delta + \phi c_{t-1} + \theta \Delta y_t + \psi y_{t-1} + u_t, \tag{2}$$

where the first-difference operator Δ is defined so that $\Delta c_t = c_t - c_{t-1}$. Estimate the first-difference model (2), and then, without using the results of (1), rederive the estimates of α, β, γ_0, and γ_1 solely on the basis of your results from (2).

Simulate model (1) of the previous question, using your estimates of α, β, γ_0, γ_1, and the variance σ^2 of the disturbances. Perform the simulation conditional on the income series and the first observation c_1 of log-consumption. Plot the residuals from running (1) on the simulated data, and compare the plot with that of the residuals from the real data. Comments?

3. Consider the following linear regression:

$$\boldsymbol{y} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u},$$

where $\boldsymbol{y}$ is $n \times 1$, $\boldsymbol{X}_1$ is $n \times k_1$, and $\boldsymbol{X}_2$ is $n \times k_2$. Let $\hat{\boldsymbol{\beta}}_1$ and $\hat{\boldsymbol{\beta}}_2$ be the OLS parameter estimates from running this regression.

Now consider the following regressions, all to be estimated by OLS:

(a) $\boldsymbol{y} = \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u}$;

(b) $\boldsymbol{P}_1\boldsymbol{y} = \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u}$;

(c) $\boldsymbol{P}_1\boldsymbol{y} = \boldsymbol{P}_1\boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u}$;

(d) $\boldsymbol{P}_{\boldsymbol{X}}\boldsymbol{y} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u}$;

(e) $\boldsymbol{P}_{\boldsymbol{X}}\boldsymbol{y} = \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u}$;

(f) $\boldsymbol{M}_1\boldsymbol{y} = \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u}$;

(g) $\boldsymbol{M}_1\boldsymbol{y} = \boldsymbol{M}_1\boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u}$;

(h) $\boldsymbol{M}_1\boldsymbol{y} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{M}_1\boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u}$;

(i) $\boldsymbol{M}_1\boldsymbol{y} = \boldsymbol{M}_1\boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{M}_1\boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u}$;

(j) $\boldsymbol{P}_{\boldsymbol{X}}\boldsymbol{y} = \boldsymbol{M}_1\boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u}$.

Here $\boldsymbol{P}_1$ projects orthogonally on to the span of $\boldsymbol{X}_1$, and $\boldsymbol{M}_1 = \mathbf{I} - \boldsymbol{P}_1$. For which of the above regressions are the estimates of $\boldsymbol{\beta}_2$ the same as for the original regression? Why? For which are the residuals the same? Why?

4. If $\boldsymbol{A}$ is a symmetric positive definite $k \times k$ matrix, then $\mathbf{I} - \boldsymbol{A}$ is positive definite if and only if $\boldsymbol{A}^{-1} - \mathbf{I}$ is positive definite, where $\mathbf{I}$ is the $k \times k$ identity matrix. Prove this result by considering the quadratic form $\boldsymbol{x}^\top(\mathbf{I} - \boldsymbol{A})\boldsymbol{x}$ and expressing $\boldsymbol{x}$ as $\boldsymbol{R}^{-1}\boldsymbol{z}$, where $\boldsymbol{R}$ is a symmetric matrix such that $\boldsymbol{A} = \boldsymbol{R}^2$.

Extend the above result to show that, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric positive definite matrices of the same dimensions, then $\boldsymbol{A} - \boldsymbol{B}$ is positive definite if and only if $\boldsymbol{B}^{-1} - \boldsymbol{A}^{-1}$ is positive definite.

5. Show that the difference between the unrestricted OLS estimator $\hat{\boldsymbol{\beta}}$ from the model

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Z\gamma} + \boldsymbol{u}, \quad \boldsymbol{u} \sim \text{IID}(\mathbf{0}, \sigma^2\mathbf{I})$$

and the OLS estimator $\tilde{\boldsymbol{\beta}}$ from the restricted model

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{u}, \quad \boldsymbol{u} \sim \text{IID}(\mathbf{0}, \sigma^2\mathbf{I})$$

is given by

$$\hat{\boldsymbol{\beta}} - \tilde{\boldsymbol{\beta}} = (\boldsymbol{X}^\top\boldsymbol{M_Z X})^{-1}\boldsymbol{X}^\top\boldsymbol{M_Z M_X y}.$$