Economics 662D1

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Assignment 2

So-called **Artificial Regressions** are often used with non-linear models, for various purposes. An essential feature of an artificial regression is that it can be run by OLS, and that its regressand and regressors are in general functions not only of data, but also of parameters. Thus, in order to run an artificial regression, parameter values must be chosen at which the regressand and regressors are evaluated.

Consider an arbitrary non-linear regression, written as follows:

$$\boldsymbol{y} = \boldsymbol{x}(\boldsymbol{\beta}) + \boldsymbol{u}. \tag{1}$$

All of the terms in the above equation are n-vectors, with $\boldsymbol{y}$ and $\boldsymbol{u}$ having their usual interpretations as dependent variable and disturbances, and $\boldsymbol{\beta}$ a k-vector of parameters to be estimated. The vector $\boldsymbol{x}(\boldsymbol{\beta})$ of regression functions is in general a *non-linear* function of parameters and data.

An artificial regression called the **Gauss-Newton regression**, or **GNR**, can be associated to the non-linear regression. The GNR looks like:

$$\boldsymbol{y} - \boldsymbol{x}(\boldsymbol{\beta}) = \boldsymbol{X}(\boldsymbol{\beta})\boldsymbol{b} + \text{residuals}. \tag{2}$$

Here $\boldsymbol{X}(\boldsymbol{\beta})$ is an $n \times k$ matrix, with typical entry

$$[\boldsymbol{X}(\boldsymbol{\beta})]_{ti} = \frac{\partial x_t(\boldsymbol{\beta})}{\partial \beta_i}, \quad t = 1, \ldots, n, \quad i = 1, \ldots k.$$

The matrix is in fact the Jacobian of the elements of $\boldsymbol{x}(\boldsymbol{\beta})$ with respect to the elements of $\boldsymbol{\beta}$. The k-vector $\boldsymbol{b}$ contains artificial parameters, and the notation + residuals indicates that the artificial regression is not an econometric model, but just a means of calculation.

Show that the OLS estimating equations for the linear regression (2) are equivalent to the first-order conditions for the minimisation of the sum of squared residuals from the non-linear regression (1). (This SSR is just $\|\boldsymbol{y} - \boldsymbol{x}(\boldsymbol{\beta})\|^2$.)

Consider an iterative procedure in which one starts from a preliminary estimate, $\boldsymbol{\beta}^{(0)}$ say. Then successive updates proceed as follows. $\boldsymbol{\beta}^{(j+1)}$ is defined by the equation

$$\boldsymbol{\beta}^{(j+1)} = \boldsymbol{\beta}^{(j)} + \boldsymbol{b}^{(j)}, \quad j = 0, 1, \ldots$$

where $\boldsymbol{b}^{(j)}$ is the k-vector of OLS estimates from the GNR (2) with the regressand and regressors evaluated at $\boldsymbol{\beta}^{(j)}$. Show that, if this iterative procedure converges at iteration m, then $\boldsymbol{\beta}^{(m)}$ satisfies the first-order conditions for the minimisation of the SSR from (1). The estimator defined in this way is called the **non-linear least squares** (**NLS**) estimator.

Conduct a simulation experiment intended to measure the bias of the NLS estimator for the non-linear regression model

$$\boldsymbol{y} = \alpha\boldsymbol{\iota} + \boldsymbol{x}_1\beta + \boldsymbol{x}_2/\beta + \boldsymbol{u}; \quad \boldsymbol{u} \sim \text{NID}(0, \sigma^2), \tag{3}$$

with $\alpha = 10$, $\beta = 1$, $\sigma = 10$. The explanatory variables can be found in the file

https://russell-davidson.arts.mcgill.ca/e662/e662.as2.21.dat

for a sample of $n = 113$ observations. For each simulation, generate $\boldsymbol{y}$ using the DGP (3), and estimate α and β by the iterative procedure given above, but never using more than 10 iterations even if convergence has not been exactly attained. Give estimates of the bias of the NLS estimates of α and β.