As for the midterm exam, do not be upset if this exam seems too long for you. Do not waste time on questions for which you do not see how to obtain the answer. Rather, answer as much as you can. Everything you do will be taken into account. Note that the different questions do not *have equal weight. However, it should be worth your while to take the time to read over the entire exam before plunging in.*

1. An economic variable $\boldsymbol{y}$, available as a series of quarterly observations, was seasonally adjusted by regression, that is, $\boldsymbol{y}$ was regressed on four seasonal dummy variables, $\boldsymbol{s}_1$, $\boldsymbol{s}_2$, $\boldsymbol{s}_3$, and $\boldsymbol{s}_4$. Denote the adjusted series by $\boldsymbol{y}'$. What are the properties of the parameter estimates from the following regressions?

$$\begin{aligned}
\boldsymbol{y}' &= \alpha + \boldsymbol{u};\\
\boldsymbol{y}' &= \beta_1\boldsymbol{s}_1 + \boldsymbol{u};\\
\boldsymbol{y}' &= \beta_1\boldsymbol{s}_1 + \beta_2\boldsymbol{s}_2 + \beta_3\boldsymbol{s}_3 + \boldsymbol{u};\\
\boldsymbol{y}' &= \alpha + \beta_1\boldsymbol{s}_1 + \beta_2\boldsymbol{s}_2 + \beta_3\boldsymbol{s}_3 + \boldsymbol{u};\\
\boldsymbol{y}' &= \alpha + \beta_1\boldsymbol{s}_1 + \beta_2\boldsymbol{s}_2 + \beta_3\boldsymbol{s}_3 + \beta_4\boldsymbol{s}_4 + \boldsymbol{u};\\
\boldsymbol{s}_1 &= \gamma\boldsymbol{y}' + \delta_2\boldsymbol{s}_2 + \boldsymbol{u};\\
\boldsymbol{y}' &= \theta\boldsymbol{x}' + \boldsymbol{u};\\
\boldsymbol{y}' &= \alpha + \theta\boldsymbol{x}' + \boldsymbol{u}.
\end{aligned}$$

The quarterly variable $\boldsymbol{x}'$ was seasonally adjusted in exactly the same way as $\boldsymbol{y}'$.

2. A variable represented by an n-vector $\boldsymbol{y}$ is said to be **centred** if the sum of its elements is zero:

$$\sum_{t=1}^{n} y_t = 0.$$

A non-centred variable $\boldsymbol{y}$ can be centred by replacing each element y_t by $y_t - \bar{y}$, where $\bar{y}$ is the mean of the elements of $\boldsymbol{y}$.
Consider the following regression:

$$\boldsymbol{y} = \alpha + \beta\boldsymbol{x} + \boldsymbol{u}. \tag{4}$$

Denote by $\boldsymbol{w}$ the result of centring $\boldsymbol{y}$, and by $\boldsymbol{z}$ the result of centring $\boldsymbol{x}$, and then consider these regressions:

$$\begin{aligned}
\boldsymbol{w} &= \beta\boldsymbol{z} + \boldsymbol{u},\\
\boldsymbol{w} &= \alpha + \beta\boldsymbol{z} + \boldsymbol{u},\\
\boldsymbol{y} &= \alpha + \beta\boldsymbol{z} + \boldsymbol{u},\\
\boldsymbol{y} &= \beta\boldsymbol{z} + \boldsymbol{u},\\
\boldsymbol{y} &= \beta\boldsymbol{x} + \boldsymbol{u}.
\end{aligned}$$

For which of these regressions is the estimate of β the same as that given by (4)? Why? In the two cases in which α is estimated, what is the connection between these estimates of α and that given by (4)? Explain.

After estimating the equation

$$\boldsymbol{y} = \alpha + \beta_1 \boldsymbol{x}_1 + \beta_2 \boldsymbol{x}_2 + \beta_3 \boldsymbol{x}_3 + \boldsymbol{u},$$

by OLS, a student observed that the regressors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$ were linearly dependent – the package had eliminated $\boldsymbol{x}_3$ on the grounds of collinearity with the other regressors. Another attempt was made using centred versions of the four variables $\boldsymbol{y}$, $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$, and eliminating the constant from the regression. Was there still a problem of collinearity? Explain. If there was, what was the linear dependency? If not, would there be a collinearity problem if the constant was put back into the regression?

3. In the following model, C_t denotes household expenditures and Y_t disposable income in period t:

$$C_t = \alpha + \beta Y_t + u_t. \tag{1}$$

Here are the results of a regression run on quarterly Canadian data, from the first quarter of 1947 to the last quarter of 1987; a sample of 164 observations:

```
ols C c Y

Ordinary Least Squares:

constant  11.390109       0.981410    11.605858
Y          0.821459       0.004901   167.624348

Number of observations = 164  Number of regressors = 2
Sum of squared residuals = 5718.656539
```

Next the constant in the above regression was replaced by four seasonal dummies, each one of which equals 1 for all the observations occurring in one of the four seasons, and 0 for the other three. The regression thus becomes

$$C_t = \sum_{i=1}^{4} \gamma_i s_{ti} + \beta Y_t + u_t, \tag{2}$$

and the results of running it are:

```
ols C S Y

Ordinary Least Squares:

Variable    Parameter estimate  Standard error  T statistic

S1           10.995811            1.269528       8.661341
S2           11.110315            1.275559       8.710151
S3           11.721576            1.280529       9.153701
S4           11.782128            1.287173       9.153493
Y             0.821389            0.004939     166.312833

  Number of observations = 164  Number of regressors = 5
  Sum of squared residuals = 5698.299247
```

Compute the F statistic that tests the null hypothesis of no seasonality (model (1)) against the seasonal alternative given by model (2). Compute estimates of the difference between autonomous consumption in each of the four seasons and the average autonomous consumption.

4. State and give a proof of the Gauss-Markov theorem concerning the efficiency of the OLS estimator of the parameters of a linear regression. Be sure to state clearly the conditions that must be satisfied in order that the conclusion of the theorem should be true. Explain clearly the sense of the term "efficiency" in the context of the theorem.

Consider the following two regressions:

$$\boldsymbol{y} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{u}, \text{ and}$$
$$\boldsymbol{y} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u}.$$

It is plain that the second regression reduces to the first if $\boldsymbol{\beta}_2 = \mathbf{0}$. Suppose that you have the results of the OLS estimation of both of these regressions. In terms of these results, how would you compute an F statistic that would let you test the hypothesis that $\boldsymbol{\beta}_2 = \mathbf{0}$? Suppose that the covariance matrix of the disturbances, $\boldsymbol{\Omega} \equiv \mathrm{Var}(\boldsymbol{u})$ is not a scalar matrix. By this is meant that there does not exist any σ^2 such that $\boldsymbol{\Omega} = \sigma^2\mathbf{I}$. Is the F statistic computed as you have described above still valid to test the hypothesis? Why or why not?

5. In terms of the standard normal ($\mathrm{N}(0,1)$) distribution, give definitions of the chi-squared (χ^2) distribution with n degrees of freedom, the Student's t distribution with n degrees of freedom, and Snedecor's F distribution with n and d degrees of freedom.

Consider the regression

$$y_t = \alpha + \beta_1 x_{t1} + \beta_2 x_{t2} + u_t, \qquad t = 1, \ldots, n. \tag{3}$$

The following printout gives (some of) the results from running this regression:

```
ols y c x1 x2

Ordinary Least Squares:

Variable        Parameter estimate      Standard error          T statistic

c               0.358382                1.990130                0.180080
x1              3.411438                0.618311                5.517351
x2              2.005344                0.632122                3.172400

Number of regressors = 3
Sum of squared residuals = 1283.780282
Explained sum of squares = 2817.408752
Estimate of residual variance (with d.f. correction) = 22.522461
R squared (uncentred) = 0.686974

Estimated covariance matrix:
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3.960618	-0.865149	0.534929
-0.865149	0.382308	0.106485
0.534929	0.106485	0.399578

What is the numerical value of n, the sample size?
Consider the hypothesis that $\beta_2 = 0$. Calculate numerically two statistics that can be used to test this hypothesis, a t statistic and an F statistic. Show that the F statistic is the square of the t statistic. Do you think that the hypothesis can be rejected by a test of level 5%?
Prove algebraically, by use of the FWL Theorem or otherwise, that the square of the t statistic and the F statistic will always be the same for the test of $\beta_2 = 0$ in (3), regardless of the actual numerical values of the variables $\boldsymbol{y}$, $\boldsymbol{x}_1$, and $\boldsymbol{x}_2$.
Give the numerical value of the estimate of $\beta_1 - \beta_2$, when the hypothesis that $\beta_2 = 0$ is not imposed. Also give the value of the estimated variance of $\beta_1 - \beta_2$. Use the values you have calculated to find the value of the t statistic for the hypothesis that $\beta_1 - \beta_2 = 0$. Do the same for the hypothesis that $\beta_1 - \beta_2 = 1$.
Suppose that $\boldsymbol{y}$ is regressed on a constant and the single nonconstant regressor $\boldsymbol{x}_1 + \boldsymbol{x}_2$. What would be the numerical value of the sum of squared residuals from this regression?

6. If the linear regression model $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{u}$ is estimated using a matrix $\boldsymbol{W} \neq \boldsymbol{X}$ of instrumental variables, the estimating equations can be written as

$$\boldsymbol{W}^\top(\boldsymbol{y} - \boldsymbol{X\beta}) = 0. \tag{4}$$

Show that the vector of fitted values from this regression is the result of acting on the dependent variable $\boldsymbol{y}$ with an oblique projection, that we can denote by $\boldsymbol{P}_{\boldsymbol{W},\boldsymbol{X}}$. Show that $\boldsymbol{P}_{\boldsymbol{W},\boldsymbol{X}}\boldsymbol{X} = \boldsymbol{X}$. Characterise the set of vectors $\boldsymbol{z}$ that are annihilated by $\boldsymbol{P}_{\boldsymbol{W},\boldsymbol{Z}}$.
Show that one can obtain the same parameter estimates as those given by (4) by first regressing the explanatory variables $\boldsymbol{X}$ on the instruments $\boldsymbol{W}$, saving the fitted values $\boldsymbol{P_W X}$, and then regressing $\boldsymbol{y}$ on $\boldsymbol{P_W X}$ by OLS.
If $\boldsymbol{X} = [\boldsymbol{X}_1 \;\; \boldsymbol{X}_2]$, this last regression can be written as

$$\boldsymbol{y} = \boldsymbol{P_W X}_1\boldsymbol{\beta}_1 + \boldsymbol{P_W X}_2\boldsymbol{\beta}_2 + \boldsymbol{u}. \tag{5}$$

Consider the null hypothesis that $\boldsymbol{\beta}_2 = \boldsymbol{0}$, which converts (5) into

$$\boldsymbol{y} = \boldsymbol{P_W X}_1\boldsymbol{\beta}_1 + \boldsymbol{u}. \tag{6}$$

Can this hypothesis be tested by an F test that treats (6) and (5) as the restricted and unrestricted model respectively? If so, why; if not, why not?

7. A confidence region for a vector of parameters $\boldsymbol{\beta}$ at a given level of confidence $1-\alpha$, $0 < \alpha < 1$, is defined as the set of parameter values $\boldsymbol{\beta}_0$ for which the hypothesis that $\boldsymbol{\beta} = \boldsymbol{\beta}_0$ cannot be rejected by some specified test at level α. Consider the parameters $\boldsymbol{\beta}$ in the following classical normal linear regression:

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Z\gamma} + \boldsymbol{u}.$$

Let the OLS estimates of $\boldsymbol{\beta}$ from this regression be denoted $\hat{\boldsymbol{\beta}}$, and the OLS estimate of the variance of the disturbances be s^2.
Show that, if the test used for hypothesis testing is the usual F test, a confidence region for $\boldsymbol{\beta}$ with confidence level $1-\alpha$ is a set of the form

$$R_\alpha = \{\boldsymbol{\beta}_0 \mid \|\boldsymbol{M_Z X}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)\|^2 \le c_\alpha s^2\}.$$

How is the constant c_α defined in terms of the CDF of the F distribution?

8. Consider a Chow test for parameter constancy in the context of the following model, which represents the alternative hypothesis:

$$\begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1\boldsymbol{\beta}_1 \\ \boldsymbol{X}_2\boldsymbol{\beta}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix}.$$

Here $\boldsymbol{X}_1$ is an $n_1 \times k$ matrix, and $\boldsymbol{X}_2$ is $n_2 \times k$, with $n_2 < k$. The null hypothesis is that $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2$. Express the alternative hypothesis as a regression augmented relative to the null-hypothesis regression. What is the distribution under the null of the F statistic computed as usual on the basis of the sums of squared residuals from the null model and the augmented regression that represents the alternative?

Suppose now that the econometrician wishes to impose the constancy of a subgroup of the parameters, while testing for the constancy of the other parameters. The alternative hypothesis now looks like

$$\begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_{11}\boldsymbol{\beta}_1 \\ \boldsymbol{X}_{21}\boldsymbol{\beta}_1 \end{bmatrix} + \begin{bmatrix} \boldsymbol{X}_{12}\boldsymbol{\beta}_{12} \\ \boldsymbol{X}_{22}\boldsymbol{\beta}_{22} \end{bmatrix} + \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix}.$$

How can an appropriate test be implemented? What is the distribution of the test statistic under the null?

9. Given an IID sample of n observations drawn from a distribution with cumulative distribution function (CDF) F, write down the expression of the empirical distribution function (EDF) of the sample.

There is a very general principle by which estimators of parameters can be defined by the same equations as the parameters themselves, after replacing all occurrences of the underlying distribution by the empirical distribution provided by some observed sample. For instance, the mean of the distribution characterised by the cumulative distribution function F is defined by the relation

$$m = \mathrm{E}_F(X),$$

and, if one has n independent drawings from this distribution, one may estimate the mean m by

$$\hat{m} \equiv \mathrm{E}_{\hat{F}}(X),$$

where the expectation is calculated using the empirical distribution function $\hat{F}$ defined by the n drawings. Show that the above definition of $\hat{m}$ simply yields the sample mean.

The same principle underlies the bootstrap method. Given that the variance of the estimator $\hat{m}$ is, by definition,

$$\mathrm{Var}(\hat{m}) = \mathrm{E}_F\big((\hat{m} - \mathrm{E}_F\hat{m})^2\big),$$

what is the bootstrap estimator of the variance of $\hat{m}$? (*Not* of m.) How is this estimator related to the usual estimator of the variance of $\hat{m}$?