Economics 468

October 15, 2019 **R. Davidson**

Midterm Examination

Do not be upset if this exam seems too long for you! Do not waste time on questions for which you do not see how to obtain the answer. Rather, answer as much as you can. Everything you do will be taken into account.

1. In a study of the consumption of Canadian households in the period from just after World War II until just before the end of the twentieth century, an econometrician had access to 196 quarterly observations on aggregate consumption, C, and aggregate disposable income, Y. The following linear regression was run

$$C_t = \alpha + \beta_1 Y_t + \beta_2 C_{t-1} + \beta_3 Y_{t-1} + u_t,$$

with the following results:

Variable	Parameter estimate	Standard error	T statistic
constant	692.534676	317.033734	2.184420
Y	0.224127	0.043389	5.165524
C1	0.995626	0.017438	57.095848
Y1	-0.217717	0.044799	-4.859841

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Number of observations = 196   Number of estimated parameters = 4
Mean of dependent variable = 192729.428571
Sum of squared residuals = 7.196138e+08
Explained sum of squares = 9.241266e+12
Estimate of residual variance
  (with d.f. correction) = 3.747988e+06
Mean of squared residuals = 3.671499e+06
Standard error of regression = 1935.972217
R squared (uncentred) = 0.999922  (centred) = 0.999633
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In an attempt to take account of the seasonality in the data, four seasonal indicator variables were constructed, s_1, s_2, s_3, s_4. When these were included in the regression, the results changed to the following.

Variable	Parameter estimate	Standard error	T statistic
s1	304.916696	398.376636	0.765398
s2	783.302973	395.228453	1.981899
s3	827.817535	396.489261	2.087869
s4	855.275634	401.594947	2.129697
Y	0.228487	0.043540	5.247729
C1	0.995487	0.017465	56.999603
Y1	-0.221988	0.044954	-4.938079

Why did the econometrician not include a constant in this regression? How can one interpret the coefficients of the four seasonal variables?

Next, three new variables were constructed:

$$x_2 = s_1 - s_2, \quad x_3 = s_1 - s_3, \quad x_4 = s_1 - s_4.$$

Another regression was run, with these three variables, and also a constant.

Variable	Parameter estimate	Standard error	T statistic
constant	692.828210	317.405267	2.182787
x2	-90.474764	239.838060	-0.377233
x3	-134.989326	240.091636	-0.562241
x4	-162.447424	239.862502	-0.677252
Y	0.228487	0.043540	5.247729
C1	0.995487	0.017465	56.999603
Y1	-0.221988	0.044954	-4.938079

Explain why the coefficients of the three economic variables, `Y`, `C1`, and `Y1`, are the same in the second and third regressions above, but not in the first. Show that the estimated constant in this last regression is the average of the coefficients of the four seasonal variables in the second regression, and explain why this is the case.

The econometrician wished to explain these results to some colleagues who were not econometricians. First, all four economic variables, `C`, `Y`, `C1`, and `Y1`, were regressed on the four seasonal variables, and the residuals from these four regressions saved as `Cs`, `Ys`, `C1s`, and `Y1s`. Then `Cs` was regressed on the other three variables, without a constant:

Variable	Parameter estimate	Standard error	T statistic
Ys	0.228487	0.043087	5.302969
C1s	0.995487	0.017283	57.599615
Y1s	-0.221988	0.044486	-4.990060

What was the interest of this last regression? Why was no constant necessary? Why were the coefficients of the economic variables equal to those in the other regressions except the first?

2. Let $\boldsymbol{P}$ and $\boldsymbol{M}$ be two complementary orthogonal projection matrices. Show that the matrix $\boldsymbol{M}-\boldsymbol{P}$ is an orthogonal matrix, by which is meant that its transpose is its inverse.

Let $\boldsymbol{u}$ and $\boldsymbol{x}$ be $n \times 1$ column vectors, and let $\boldsymbol{P} = \boldsymbol{u}(\boldsymbol{u}^\top\boldsymbol{u})^{-1}\boldsymbol{u}^\top$. Express $\boldsymbol{u}$ and $\boldsymbol{x}$ as follows, separating the first elements of each from the others:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \boldsymbol{x}_2 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{u} = \begin{bmatrix} u_1 \\ \boldsymbol{u}_2 \end{bmatrix}.$$

It is wished to select $\boldsymbol{u}$ in such a way that all elements of $(\boldsymbol{M} - \boldsymbol{P})\boldsymbol{x}$ except the first are zero. That is, we want

$$(\boldsymbol{M} - \boldsymbol{P})\boldsymbol{x} = \begin{bmatrix} s_1 \\ \boldsymbol{0} \end{bmatrix},$$

for some s_1. Show that this can be done by setting $\boldsymbol{u}_2 = \boldsymbol{x}_2$, with an appropriate choice of u_1. Show that this appropriate choice could be either of the quantities $x_1 \pm \sqrt{x_1^2 + \|\boldsymbol{x}_2\|^2}$.

For those who may be interested, this operation is called a **Householder reflection**. Here, Householder is a proper name, not a generic economic agent!

3. Consider the linear regression model $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{u}$, and suppose that the true DGP* is $\boldsymbol{y} = \boldsymbol{X\beta}_0 + \boldsymbol{u}$, with $\mathrm{E}(\boldsymbol{u}|\boldsymbol{X}) = \mathbf{0}$ and $\mathrm{E}(\boldsymbol{uu}^\top|\boldsymbol{X}) = \sigma_0^2\mathbf{I}$, for some variance parameter σ_0^2, and with $\mathbf{I}$ the $n \times n$ identity matrix. We suppose that $\boldsymbol{X}$ is an $n \times k$ matrix, and that $\boldsymbol{W}$ is an $n \times k$ matrix of instrumental variables such that $\mathrm{E}(\boldsymbol{u}|\boldsymbol{W}) = \mathbf{0}$ and $\mathrm{E}(\boldsymbol{uu}^\top|\boldsymbol{W}) = \sigma_0^2\mathbf{I}$. Suppose that $\boldsymbol{W}^\top\boldsymbol{X}$ is a $k \times k$ non-singular matrix.

Show that the estimators $\hat{\boldsymbol{\beta}}_{\boldsymbol{W}}$ and $\hat{\boldsymbol{\beta}}_{\boldsymbol{X}}$ given by the estimating equations

$$\boldsymbol{W}^\top(\boldsymbol{y} - \boldsymbol{X\beta}) = \mathbf{0} \quad \text{and} \quad \boldsymbol{X}^\top(\boldsymbol{y} - \boldsymbol{X\beta}) = \mathbf{0},$$

respectively, are both unbiased.

If $\boldsymbol{P_X}$ and $\boldsymbol{M_X}$ are the complementary orthogonal projections that project respectively on to the linear span of the columns of $\boldsymbol{X}$ and the orthogonal complement of that linear span, we have $\boldsymbol{P_X} + \boldsymbol{M_X} = \mathbf{I}$. The first of the estimating equations above can therefore be written as

$$\boldsymbol{W}^\top(\boldsymbol{P_X} + \boldsymbol{M_X})(\boldsymbol{y} - \boldsymbol{X\beta}) = \mathbf{0}.$$

Show that this implies that

$$\boldsymbol{W}^\top\boldsymbol{X}(\hat{\boldsymbol{\beta}}_{\boldsymbol{W}} - \hat{\boldsymbol{\beta}}_{\boldsymbol{X}}) = \boldsymbol{W}^\top\boldsymbol{M_X y}.$$

Show that this in turn implies that the $k \times 1$ vector $\hat{\boldsymbol{\beta}}_{\boldsymbol{W}} - \hat{\boldsymbol{\beta}}_{\boldsymbol{X}}$ is uncorrelated with $\hat{\boldsymbol{\beta}}_{\boldsymbol{X}}$. Why does this imply the Gauss-Markov theorem, according to which $\boldsymbol{\beta_X}$ is BLUE (Best Linear Unbiased Estimator)? Here, this means that the difference between the covariance matrices of $\hat{\boldsymbol{\beta}}_{\boldsymbol{W}}$ and $\hat{\boldsymbol{\beta}}_{\boldsymbol{X}}$ is a positive semi-definite matrix.

4. Suppose that the $n \times k$ matrix $\boldsymbol{X}$ of explanatory variables is partitioned into two submatrices, as follows: $\boldsymbol{X} = [\boldsymbol{X}_1 \quad \boldsymbol{X}_2]$, where $\boldsymbol{X}_1$ has k_1 columns and $\boldsymbol{X}_2$ has k_2 columns. If the orthogonal projections on to the spans of the columns of $\boldsymbol{X}$ and $\boldsymbol{X}_1$ are denoted by $\boldsymbol{P_X}$ and $\boldsymbol{P}_1$ respectively, show that $\boldsymbol{P_X} - \boldsymbol{P}_1$ is an orthogonal projection matrix.
Show that the trace (that is, the sum of the diagonal elements) of $\boldsymbol{P_X}$ is equal to k. What is the trace of $\boldsymbol{P_X} - \boldsymbol{P}_1$?
Show that any $n \times 1$ vector $\boldsymbol{z}$ of the form $\boldsymbol{M}_1\boldsymbol{X}_2\boldsymbol{\gamma}$, for an arbitrary $k_2 \times 1$ vector $\boldsymbol{\gamma}$, is left unchanged when premultiplied by $\boldsymbol{P_X} - \boldsymbol{P}_1$; that is, show that $(\boldsymbol{P_X} - \boldsymbol{P}_1)\boldsymbol{M}_1\boldsymbol{X}_2\boldsymbol{\gamma} = \boldsymbol{M}_1\boldsymbol{X}_2\boldsymbol{\gamma}$.
Why do the above results prove that $\boldsymbol{P_X} - \boldsymbol{P}_1 = \boldsymbol{P}_{\boldsymbol{M}_1\boldsymbol{X}_2}$, where this last matrix denotes the orthogonal projection on to $\mathcal{S}(\boldsymbol{M}_1\boldsymbol{X}_2)$?

* Data-generating process, in case you have forgotten.