McGill

June 2017
Final Examination

Models for Financial Economics
Economics 765

Take-home exam due on or before Saturday June 24th 2017.

This exam comprises 6 pages, including the cover page

1. Exercise 1.15 in Shreve. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$, and assume that X has a density function $f(x)$ that is positive for all $x \in \mathbb{R}$. Let g be a strictly increasing differentiable function satisfying

$$\lim_{y \to -\infty} g(y) = -\infty, \qquad \lim_{y \to \infty} g(y) = \infty,$$

and define the random variable $Y = g(X)$.
Let $h(Y)$ be an arbitrary non-negative function satisfying $\int_{-\infty}^{\infty} h(y)\,\mathrm{d}y = 1$. We want to change the probability measure so that $h(y)$ is the density function for the random variable Y. To do this, we define

$$Z = \frac{h\bigl(g(X)\bigr)g'(X)}{f(X)}.$$

(i) Show that Z is non-negative and that $\mathrm{E}Z = 1$.

Now define $\widetilde{P}$ by

$$\widetilde{P}(A) = \int_A Z \,\mathrm{d}P \qquad \text{for all } A \in \mathcal{F}.$$

(ii) Show that Y has density h under $\widetilde{P}$.

2. In his Theorem 3.7.3, Shreve shows that the joint density of the pair $(M(t), W(t))$ is

$$f_{M(t),W(t)}(m, w) = \frac{2(2m - w)}{t\sqrt{2\pi t}} \exp -\frac{(2m - w)^2}{2t}, \qquad w \le m, m > 0,$$

where $W(t)$ is standard Brownian motion, and

$$M(t) = \max_{0 \le s \le t} W(s)$$

is the maximum to date of the Brownian motion.
Consider a Brownian motion with drift. Let

$$\hat{W}(t) = \alpha t + W(t),$$

and the corresponding maximum to date

$$\hat{M}(t) = \max_{0 \le s \le t} \hat{W}(t).$$

Use Girsanov's theorem to change the measure in such a way that $\hat{W}(t)$ has no drift under the new measure. Change back to the original measure and show that, under it, the joint density of the pair $(\hat{M}(t), \hat{W}(t))$ is

$$\hat{f}_{\hat{M}(t),\hat{W}(t)}(m, w) = \frac{2(2m - w)}{t\sqrt{2\pi t}} \exp\Bigl[\alpha w - \tfrac{1}{2}\alpha^2 t - \frac{(2m - w)^2}{2t}\Bigr], \qquad w \le m, m > 0.$$

3. Let $(\Omega, \mathcal{F}, P)$ be a probability space with Ω the unit interval $[0,1]$, $\mathcal{F}$ the Borel σ-algebra $\mathcal{B}$, and P Lebesgue measure. Let $\mathcal{F}_n$ be the σ-algebra generated by the intervals of the form $\big[(j-1)2^{-n}, j2^{-n}\big]$, $j = 1, 2, \ldots, 2^n$, open to the left, closed to the right. Let X be a bounded continuous function on $[0,1]$.

(i) Give the explicit form of the conditional expectation $\mathrm{E}(X \,|\, \mathcal{F}_n)$.

(ii) Show that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all n.

(iii) Show that the sequence of random variables $\{\mathrm{E}(X \,|\, \mathcal{F}_n)\}$, $n = 1, 2, \ldots$, converges for almost all $\omega \in [0,1]$. Characterise the limiting variable. **Hint:** Express ω as the infinite sum

$$\omega = \sum_{j=1}^{\infty} \omega_j 2^{-j}$$

where $\omega_j = 0$ or 1 for all j. Each sequence $\{\omega_j\}$ defines a unique real number ω, and each real $\omega \in [0,1]$ defines a unique sequence, provided that one excludes sequences such that there exists $J > 0$ finite with $\omega_J = 1$ and $\omega_j = 0$ for all $j > J$.

(iv) Show directly that $\mathrm{E}\big(\mathrm{E}(X \,|\, \mathcal{F}_n)\big) = \mathrm{E}(X)$.

4. Let X be a real-valued random variable defined on the probability space $(\Omega, \mathcal{F}, P)$.

(i) Suppose that $\mathrm{E}|X|^p$ exists, where $p > 0$. Prove the *Markov inequality*, which states that, for all $\varepsilon > 0$,

$$P(|X| > \varepsilon) \le \frac{\mathrm{E}|X|^p}{\varepsilon^p}. \tag{1}$$

(ii) Let $g : \mathbb{R} \to \mathbb{R}_+$ be an increasing function. Show that, for all real a,

$$P(X > a) \le \frac{\mathrm{E}(g(X))}{g(a)}. \tag{2}$$

(iii) Suppose that a discrete-time filtration is defined on the measure space $(\Omega, \mathcal{F})$ as a nested set of σ-algebras $\mathcal{F}_t$, $t = 0, 1, 2, \ldots$, with $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$. A martingale in this discrete-time context is a stochastic process X_t, defined for $t = 0, 1, \ldots$, and such that X_t is $\mathcal{F}_t$–measurable for all t, and

$$\mathrm{E}\big(X_{t+1} \,|\, \mathcal{F}_t\big) = X_t. \tag{3}$$

Show that (3) implies that, for all $t = 0, 1, \ldots$,

$$\mathrm{E}\big(X_{t+s} \,|\, \mathcal{F}_t\big) = X_t, \quad s = 1, 2, \ldots. \tag{4}$$

(iv) A stopping time τ relative to the filtration $\mathcal{F}_t$ is a random variable that takes on the possible values $0, 1, 2, \ldots, \infty$. It is such that the event $\{\omega : \tau(\omega) = k\}$

belongs to $\mathcal{F}_k$ for all $k = 0, 1, \ldots$. The stopped process $X_{t\wedge\tau}$ is defined by the equation

$$X_{t\wedge\tau}(\omega) = X_{\min(t,\tau(\omega))}(\omega).$$

Show that

$$X_{t\wedge\tau} = \sum_{s=1}^{t-1} \mathrm{I}(\tau = s)X_s + \mathrm{I}(\tau \geq t)X_t.$$

If X_t is a martingale, show that the process $X_{t\wedge\tau}$ is also a martingale.

(v) The σ-algebra $\mathcal{F}_\tau$ defined by a stopping time τ is such that $A \in \mathcal{F}_\tau$ iff the event $A \cap \{\tau = t\} \in \mathcal{F}_t$ for all $t = 0, 1, 2, \ldots$. If $\tau < n$ almost surely for some finite positive integer n, show that, for all $t = 0, 1, \ldots, n$,

$$\mathrm{E}(X_t \,|\, \mathcal{F}_\tau) = \sum_{s=0}^{n} \mathrm{E}\big(X_t \mathrm{I}(\tau = s) \,|\, \mathcal{F}_s\big).$$

5. Shreve's Exercise 6.7 on the Heston stochastic volatility model is very long. You should refer to it for details that may be helpful but are omitted here.
A stock price has the usual differential based on geometric Brownian motion:

$$\mathrm{d}S(t) = rS(t)\,\mathrm{d}t + \sqrt{V(t)}S(t)\,\mathrm{d}\widetilde{W}_1(t),$$

where the interest rate r is constant, and the volatility $\sqrt{V(t)}$ is itelf a stochastic process governed by the equation

$$\mathrm{d}V(t) = \big(a - bV(t)\big)\,\mathrm{d}t + \sigma\sqrt{V(t)}\,\mathrm{d}\widetilde{W}_2(t).$$

The parameters a, b, and σ are positive constants, and $\widetilde{W}_1(t)$ and $\widetilde{W}_2(t)$ are correlated Brownian motions under a risk-neutral measure $\widetilde{P}$, with

$$\mathrm{d}\widetilde{W}_1(t)\,\mathrm{d}\widetilde{W}_2(t) = \rho\,\mathrm{d}t.$$

The two-dimensional process $\big(S(t), V(t)\big)$ is a Markov process.
At time t, the price of a European call expiring at $T > t$ is given by a function $c(t, s, v)$ such that

$$c\big(t, S(t), V(t)\big) = \widetilde{\mathrm{E}}\Big[\mathrm{e}^{-r(T-t)}\big(S(T) - K\big)_+ \,\Big|\, \mathcal{F}(t)\Big], \quad 0 \leq t \leq T.$$

We are to show that c satisfies the partial differential equation

$$c_t + rsc_s + (a - bv)c_v + \tfrac{1}{2}s^2vc_{ss} + \rho\sigma svc_{sv} + \tfrac{1}{2}\sigma^2vc_{vv} = rc \tag{5}$$

in the region $0 \leq t < T$, $s \geq 0$, $v \geq 0$, and that it satisfies the boundary condition

$$c(T, s, v) = (s - K)_+ \quad \text{for all } s \geq 0,\ v \geq 0. \tag{6}$$

Shreve suggests that we should proceed as follows.

(i) Show that $\mathrm{e}^{-rt}c\big(t, S(t), V(t)\big)$ is a martingale under $\widetilde{P}$, and then derive the PDE (5) from this fact.

(ii) Suppose that there are functions $f(t,x,v)$ and $g(t,x,v)$ satisfying

$$f_t + (r + \tfrac{1}{2}v)f_x + (a - bv + \rho\sigma v)f_v + \tfrac{1}{2}vf_{xx} + \rho\sigma v f_{xv} + \tfrac{1}{2}\sigma^2 v f_{vv} = 0, \quad (7)$$
$$g_t + (r - \tfrac{1}{2}v)g_x + (a - bv)g_v + \tfrac{1}{2}vg_{xx} + \rho\sigma v g_{xv} + \tfrac{1}{2}\sigma^2 v g_{vv} = 0, \quad (8)$$

in the region $0 \le t < T$, $-\infty < x < \infty$, and $v \ge 0$. Show that, if we define

$$c(t,s,v) = sf(t, \log s, v) - \mathrm{e}^{-r(T-t)}Kg(t, \log s, v), \quad (9)$$

then $c(t,s,v)$ satisfies (5).

(iii) Suppose a pair of processes $\big(X(t), V(t)\big)$ is governed by the stochastic differential equations

$$\mathrm{d}X(t) = \big(r + \tfrac{1}{2}V(t)\big)\,\mathrm{d}t + \sqrt{V(t)}\,\mathrm{d}W_1(t),$$
$$\mathrm{d}V(t) = \big(a - bV(t) + \rho\sigma V(t)\big)\,\mathrm{d}t + \sigma\sqrt{V(t)}\,\mathrm{d}W_2(t),$$

where $W_1(t)$ and $W_2(t)$ are Brownian motions under a measure P with $\mathrm{d}W_1(t)\,\mathrm{d}W_2(t) = \rho\mathrm{d}t$. Define

$$f(t,x,v) = \mathrm{E}^{t,x,v}\mathrm{I}\big(X(T) \ge \log K\big).$$

Show that $f(t,x,v)$ satisfies (7) and the boundary condition

$$f(T,x,v) = \mathrm{I}(x \ge \log K) \quad \text{for all } x \in \mathbb{R},\ v \ge 0.$$

(iv) Suppose a pair of processes $\big(X(t), V(t)\big)$ is governed by the stochastic differential equations

$$\mathrm{d}X(t) = \big(r - \tfrac{1}{2}V(t)\big)\,\mathrm{d}t + \sqrt{V(t)}\,\mathrm{d}W_1(t),$$
$$\mathrm{d}V(t) = \big(a - bV(t)\big)\,\mathrm{d}t + \sigma\sqrt{V(t)}\,\mathrm{d}W_2(t),$$

where $W_1(t)$ and $W_2(t)$ are Brownian motions under a measure P with $\mathrm{d}W_1(t)\,\mathrm{d}W_2(t) = \rho\mathrm{d}t$. Define

$$g(t,x,v) = \mathrm{E}^{t,x,v}\mathrm{I}\big(X(T) \ge \log K\big).$$

Show that $g(t,x,v)$ satisfies (8) and the boundary condition

$$g(T,x,v) = \mathrm{I}\big(x \ge \log K\big) \quad \text{for all } x \in \mathbb{R},\ v \ge 0.$$

(v) Show that with $f(t,x,v)$ and $g(t,x,v)$ as in (iii) and (iv), the function $c(t,x,v)$ of (9) satisfies the boundary condition (6).

6. This is Shreve's Exercise 8.5, about a perpetual American put that pays dividends. Consider a perpetual American put written on a geometric Brownian motion asset price paying dividends at a constant rate $a > 0$. The differential of this asset price is

$$\mathrm{d}S(t) = (r - a)S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}\widetilde{W}(t),$$

where $\widetilde{W}(t)$ is a Brownian motion under the risk-neutral measure $\widetilde{P}$.

(i) Suppose we adopt the strategy of exercising the put the first time the asset price is at or below L. What is the risk-neutral expected discounted payoff of this strategy? Write this as a function $v_L(x)$ of the initial asset price x. (**Hint:** Define the positive constant

$$\gamma = \frac{1}{\sigma^2}\Big(r - a - \frac{1}{2}\sigma^2\Big) + \frac{1}{\sigma}\sqrt{\frac{1}{\sigma^2}\Big(r - a - \frac{1}{2}\sigma^2\Big)^2 + 2r}$$

and write $v_L(x)$ using γ.)

(ii) Determine L_*, the value of L that maximises the risk-neutral expected discounted payoff computed in (i).

(iii) Show that, for any initial asset price $S(0) = x$, the process $\mathrm{e}^{-rt}v_{L_*}(S(t))$ is a supermartingale under $\widetilde{P}$. Show that if $S(0) = x > L_*$ and $\mathrm{e}^{-rt}v_{L_*}(S(t))$ is stopped the first time the asset price reaches L_*, then the stopped supermartingale is a martingale. (**Hint:** Show that

$$r + (r - a)\gamma - \frac{1}{2}\sigma^2\gamma(\gamma + 1) = 0.)$$

(iv) Show that, for any initial asset price $S(0) = x$,

$$v_{L_*}(x) = \max_{\tau \in \mathcal{T}} \widetilde{\mathrm{E}}\big[\mathrm{e}^{-r\tau}(K - S(\tau))\big],$$

where τ is a stopping time realised in the non-negative real line.

7. This is taken from Shreve's Exercises 11.3 and 11.7. Let $N(t)$ be a (simple) Poisson process with intensity λ, and let $S(0) > 0$ and $\sigma > -1$ be given. Then using just the independence of non-overlapping increments of the process N amd the distribution of these increments, show that the Geometric Poisson process

$$S(t) = \exp\{N(t)\log(\sigma + 1) - \lambda\sigma t\} = (\sigma + 1)^{N(t)}\mathrm{e}^{-\lambda\sigma t}$$

is a martingale.

Again using only the properties of Poisson increments, prove that a compound Poisson processs is a Markov process. That is, show that, whenever we are given two times $0 \le t \le T$ and a function $h(x)$, there is another function $g(t, x)$ such that

$$\mathrm{E}\big[h(Q(T))\,|\,\mathcal{F}(t)\big] = g(t, Q(t)).$$