# Economics 765

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## Assignment 5

You are asked to do exercises 6.1, 6.6, 6.8, and 6.9 of Volume 2 of Shreve. The essence of these exercises is reproduced below for convenience.

**6.1** Consider the stochastic differential equation

$$\mathrm{d}X(u) = \big(a(u) + b(u)X(u)\big)\,\mathrm{d}u + \big(\gamma(u) + \sigma(u)X(u)\big)\,\mathrm{d}W(u), \tag{6.2.4}$$

where $W(u)$ is a Brownian motion relative to a filtration $\mathcal{F}(u)$, $u \geq 0$, and we allow $a(u)$, $b(u)$, $\gamma(u)$, and $\sigma(u)$ to be processes adapted to this filtration. Fix an initial time $t$ and an initial position $x \in \mathbb{R}$. Define

$$Z(u) = \exp\left\{\int_t^u \sigma(v)\,\mathrm{d}W(v) + \int_t^u \Big(b(v) - \frac{1}{2}\sigma^2(v)\Big)\,\mathrm{d}v\right\},$$

$$Y(u) = x + \int_t^u \frac{a(v) - \sigma(v)\gamma(v)}{Z(v)}\,\mathrm{d}v + \int_t^u \frac{\gamma(v)}{Z(v)}\,\mathrm{d}W(v).$$

(i) Show that $Z(t) = 1$ and

$$\mathrm{d}Z(u) = b(u)Z(u)\,\mathrm{d}u + \sigma(u)Z(u)\,\mathrm{d}W(u), \quad u \geq t.$$

(ii) By its very definition, $Y(u)$ satisfies $Y(t) = x$ and

$$\mathrm{d}Y(u) = \frac{a(u) - \sigma(u)\gamma(u)}{Z(u)}\,\mathrm{d}u + \frac{\gamma(u)}{Z(u)}\,\mathrm{d}W(u), \quad u \geq t.$$

Show that $X(u) = Y(u)Z(u)$ solves the stochastic differential equation (6.2.4) and satisfies the initial condition $X(t) = x$.

**6.6** Problem on the moment-generating function for the Cox-Ingersoll-Ross process.

(i) Let $W_1, \ldots, W_d$ be independent Brownian motions and let $a$ and $\sigma$ be positive constants. For $j = 1, \ldots, d$, let $X_j(t)$ be the solution of the *Ornstein-Uhlenbeck* stochastic differential equation

$$\mathrm{d}X_j(t) = -\frac{b}{2}X_j(t)\,\mathrm{d}t + \frac{1}{2}\sigma\,\mathrm{d}W_j(t).$$

Show that

$$X_j(t) = \mathrm{e}^{-\frac{1}{2}bt}\left[X_j(0) + \frac{\sigma}{2}\int_0^t \mathrm{e}^{\frac{1}{2}bu}\,\mathrm{d}W_j(u)\right].$$

Show further that, for fixed $t$, the random variable $X_j(t)$ is normal with

$$\mathrm{E}X_j(t) = \mathrm{e}^{-\frac{1}{2}bt}X_j(0), \quad \mathrm{Var}(X_j(t)) = \frac{\sigma^2}{4b}\Big[1 - \mathrm{e}^{-bt}\Big].$$

(Hint: Use Theorem 4.4.9.)

(ii) Define

$$R(t) = \sum_{j=1}^{d} X_j^2(t), \tag{6.9.18}$$

and show that

$$\mathrm{d}R(t) = \big(a - bR(t)\big)\,\mathrm{d}t + \sigma\sqrt{R(t)}\,\mathrm{d}B(t),$$

where $a = d\sigma^2/4$ and

$$B(t) = \sum_{j=1}^{d} \int_0^t \frac{X_j(s)}{\sqrt{R(s)}}\,\mathrm{d}W_j(s)$$

is a Brownian motion. In other words, $R(t)$ is a Cox-Ingersoll-Ross interest rate process. (Hint: Lévy's theorem to show that $B(t)$ is a Brownian motion.)

(iii) Suppose $R(0) > 0$ given, and define

$$X_j(0) = \sqrt{\frac{R(0)}{d}}.$$

Show then that $X_1(t), \ldots, X_d(t)$ are independent, identically distributed, normal random variables, each having expectation

$$\mu(t) = \mathrm{e}^{-\frac{1}{2}bt}\sqrt{\frac{R(0)}{d}}$$

and variance

$$v(t) = \frac{\sigma^2}{4b}\big[1 - \mathrm{e}^{-bt}\big].$$

(iv) Show that the moment-generating function of the square of a variable $X$ with the $\mathrm{N}(\mu, \sigma^2)$ distribution is

$$\mathrm{E}\exp uX^2 = \frac{1}{\sqrt{1 - 2u\sigma^2}}\exp\left\{\frac{u\mu^2}{1 - 2u\sigma^2}\right\}$$

for $u < 1/2\sigma^2$.

(v) Part (iii) shows that $R(t)$ given by (6.9.18) is the sum of squares of IID normal random variables and hence has a *noncentral* $\chi^2$ *distribution*, the term "noncentral" referring to the fact that $\mu(t) = \mathrm{E}X_j(t)$ is not zero. Show that the moment-generating function of $R(t)$ is

$$\mathrm{E}\exp uR(t) = \left(\frac{1}{1 - 2v(t)u}\right)^{2a/\sigma^2}\exp\left\{\frac{\mathrm{e}^{-bt}uR(0)}{1 - 2v(t)u}\right\}$$

for $u < 1/(2v(t))$.

**6.8** Consider the SDE

$$\mathrm{d}X(u) = \beta(u, X(u))\,\mathrm{d}u + \gamma(u, X(u))\,\mathrm{d}W(u).$$

We assume that if we begin a process at an arbitrary initial positive value $X(t) = x$ at an arbitrary initial time $t$, and let it evolve according to the SDE, its value at each time $T > t$ could be any positive number but cannot be nonpositive. For $0 \leq t < T$, let $p(t, T, x, y)$ be the transition density for the solution to this equation, by which we mean that the random variable $X(T)$ that evolves from $X(t) = x$ has density $p(t, T, x, y)$ in the $y$ variable. We assume that $p(t, T, x, y) = 0$ for $0 \leq t < T$ and $y \leq 0$.

Show that $p(t, T, x, y)$ satisfies the *Kolmogorov backward equation*

$$-p_t(t, T, x, y) = \beta(t, x)p_x(t, T, x, y) + \tfrac{1}{2}\gamma^2(t, x)p_{xx}(t, T, x, y).$$

**6.9** With the same setup as for the previous exercise, show that $p(t, T, x, y)$ satisfies the *Kolmogorov forward equation*, which we may write as

$$\frac{\partial}{\partial T}p(t, T, x, y) = -\frac{\partial}{\partial y}\big(\beta(T, y)p(t, T, x, y)\big) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\gamma^2(T, y)p(t, T, x, y)\big).$$

In contrast to the Kolmogorov backward equation, in which $T$ and $y$ are held constant and the variables are $t$ and $x$, here $t$ and $x$ are held constant and the variables are $T$ and $y$.

Notes on this exercise:

- There is a misprint in Shreve in the forward equation: He writes $\beta(t, y)$ for $\beta(T, y)$.
- In the book, there is a long series of steps that can guide you to the desired result. You are at liberty to make use of this or not as you see fit.
- The partial differential operator in the forward equation is the *adjoint* of that in the backward equation. The latter is of course simpler to use.
- The forward equation is also referred to as the *Fokker-Planck equation*, after two eminent physicists.