# Chapter 14

# The Itô Integral

The following chapters deal with *Stochastic Differential Equations in Finance*. References:

1. B. Oksendal, *Stochastic Differential Equations*, Springer-Verlag,1995
2. J. Hull, *Options, Futures and other Derivative Securities,* Prentice Hall, 1993.

## 14.1 Brownian Motion

(See Fig. 13.3.) $(\Omega, \mathcal{F}, \mathbb{P})$ is given, always in the background, even when not explicitly mentioned. **Brownian motion**, $B(t,\omega) : [0,\infty) \times \Omega \to I\!R$, has the following properties:

1. $B(0) = 0$; Technically, $I\!P\{\omega; B(0,\omega) = 0\} = 1$,
2. $B(t)$ is a continuous function of $t$,
3. If $0 = t_0 \le t_1 \le \ldots \le t_n$, then the increments

   $$B(t_1) - B(t_0),\ \ldots,\ B(t_n) - B(t_{n-1})$$

   are *independent*,*normal*, and

   $$I\!E[B(t_{k+1}) - B(t_k)] = 0,$$
   $$I\!E[B(t_{k+1}) - B(t_k)]^2 = t_{k+1} - t_k.$$

## 14.2 First Variation

Quadratic variation is a measure of volatility. First we will consider *first variation*, $FV(f)$, of a function $f(t)$.

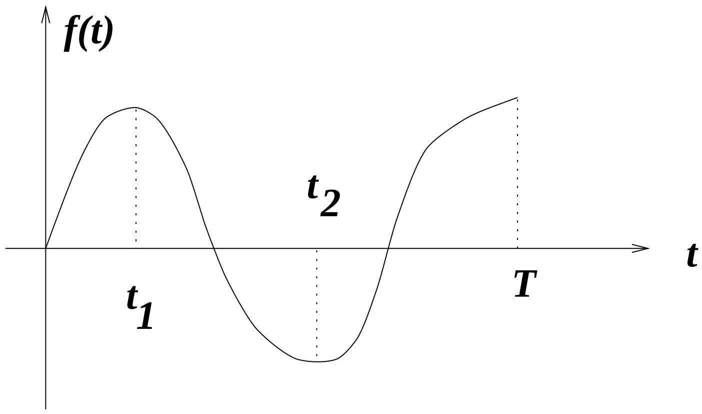


Figure 14.1: *Example function* $f(t)$.

For the function pictured in Fig. 14.1, the first variation over the interval $[0,T]$ is given by:

$$\begin{aligned} FV_{[0,T]}(f) &= [f(t_1) - f(0)] - [f(t_2) - f(t_1)] + [f(T) - f(t_2)] \\ &= \int_0^{t_1} f'(t)\,dt + \int_{t_1}^{t_2} (-f'(t))\,dt + \int_{t_2}^{T} f'(t)\,dt. \\ &= \int_0^T |f'(t)|\,dt. \end{aligned}$$

Thus, first variation measures the total amount of up and down motion of the path.

The general definition of first variation is as follows:

**Definition 14.1 (First Variation)** Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a *partition* of $[0,T]$, i.e.,

$$0 = t_0 \le t_1 \le \ldots \le t_n = T.$$

The *mesh* of the partition is defined to be

$$||\Pi|| = \max_{k=0,\ldots,n-1} (t_{k+1} - t_k).$$

We then define

$$FV_{[0,T]}(f) = \lim_{||\Pi|| \to 0} \sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)|.$$

Suppose $f$ is differentiable. Then the Mean Value Theorem implies that in each subinterval $[t_k, t_{k+1}]$, there is a point $t_k^*$ such that

$$f(t_{k+1}) - f(t_k) = f'(t_k^*)(t_{k+1} - t_k).$$

Then

$$\sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)| = \sum_{k=0}^{n-1} |f'(t_k^*)|(t_{k+1} - t_k),$$

and

$$FV_{[0,T]}(f) = \lim_{||\Pi|| \to 0} \sum_{k=0}^{n-1} |f'(t_k^*)|(t_{k+1} - t_k)$$
$$= \int_0^T |f'(t)| \, dt.$$

## 14.3 Quadratic Variation

**Definition 14.2 (Quadratic Variation)** The *quadratic variation* of a function $f$ on an interval $[0, T]$ is

$$\langle f \rangle(T) = \lim_{||\Pi|| \to 0} \sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)|^2.$$

**Remark 14.1 (Quadratic Variation of Differentiable Functions)** If $f$ is differentiable, then $\langle f \rangle(T) = 0$, because

$$\sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)|^2 = \sum_{k=0}^{n-1} |f'(t_k^*)|^2 (t_{k+1} - t_k)^2$$
$$\leq ||\Pi||. \sum_{k=0}^{n-1} |f'(t_k^*)|^2 (t_{k+1} - t_k)$$

and

$$\langle f \rangle(T) \leq \lim_{||\Pi|| \to 0} ||\Pi||. \lim_{||\Pi|| \to 0} \sum_{k=0}^{n-1} |f'(t_k^*)|^2 (t_{k+1} - t_k)$$
$$= \lim_{||\Pi|| \to 0} ||\Pi|| \int_0^T |f'(t)|^2 \, dt$$
$$= 0.$$

**Theorem 3.44**

$$\langle B \rangle(T) = T,$$

*or more precisely,*

$$I\!\!P\{\omega \in \Omega; \langle B(., \omega) \rangle(T) = T\} = 1.$$

*In particular, the paths of Brownian motion are not differentiable.*

**Proof:** (Outline) Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0, T]$. To simplify notation, set $D_k = B(t_{k+1}) - B(t_k)$. Define the *sample quadratic variation*

$$Q_\Pi = \sum_{k=0}^{n-1} D_k^2.$$

Then

$$Q_\Pi - T = \sum_{k=0}^{n-1} [D_k^2 - (t_{k+1} - t_k)].$$

We want to show that

$$\lim_{||\Pi|| \to 0} (Q_\Pi - T) = 0.$$

Consider an individual summand

$$D_k^2 - (t_{k+1} - t_k) = [B(t_{k+1}) - B(t_k)]^2 - (t_{k+1} - t_k).$$

This has expectation 0, so

$$\mathbb{E}(Q_\Pi - T) = \mathbb{E} \sum_{k=0}^{n-1} [D_k^2 - (t_{k+1} - t_k)] = 0.$$

For $j \neq k$, the terms

$$D_j^2 - (t_{j+1} - t_j) \quad \text{and} \quad D_k^2 - (t_{k+1} - t_k)$$

are independent, so

$$\begin{aligned}
\operatorname{var}(Q_\Pi - T) &= \sum_{k=0}^{n-1} \operatorname{var}[D_k^2 - (t_{k+1} - t_k)] \\
&= \sum_{k=0}^{n-1} \mathbb{E}[D_k^4 - 2(t_{k+1} - t_k) D_k^2 + (t_{k+1} - t_k)^2] \\
&= \sum_{k=0}^{n-1} [3(t_{k+1} - t_k)^2 - 2(t_{k+1} - t_k)^2 + (t_{k+1} - t_k)^2] \\
&\qquad \text{(if } X \text{ is normal with mean 0 and variance } \sigma^2\text{, then } \mathbb{E}(X^4) = 3\sigma^4\text{)} \\
&= 2 \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2 \\
&\le 2||\Pi|| \sum_{k=0}^{n-1} (t_{k+1} - t_k) \\
&= 2||\Pi||\, T.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\mathbb{E}(Q_\Pi - T) &= 0, \\
\operatorname{var}(Q_\Pi - T) &\le 2||\Pi||.T.
\end{aligned}$$

As $||\Pi||\to 0$, $\mathrm{var}(Q_\Pi - T)\to 0$, so

$$\lim_{||\Pi||\to 0}(Q_\Pi - T) = 0.$$

■

**Remark 14.2 (Differential Representation)** We know that

$$\mathbb{E}[(B(t_{k+1}) - B(t_k))^2 - (t_{k+1} - t_k)] = 0.$$

We showed above that

$$\mathrm{var}[(B(t_{k+1}) - B(t_k))^2 - (t_{k+1} - t_k)] = 2(t_{k+1} - t_k)^2.$$

When $(t_{k+1} - t_k)$ is small, $(t_{k+1} - t_k)^2$ is *very* small, and we have the approximate equation

$$(B(t_{k+1}) - B(t_k))^2 \simeq t_{k+1} - t_k,$$

which we can write informally as

$$dB(t)\ dB(t) = dt.$$

## 14.4 Quadratic Variation as Absolute Volatility

On any time interval $[T_1, T_2]$, we can sample the Brownian motion at times

$$T_1 = t_0 \le t_1 \le \ldots \le t_n = T_2$$

and compute the *squared sample absolute volatility*

$$\frac{1}{T_2 - T_1}\sum_{k=0}^{n-1}(B(t_{k+1}) - B(t_k))^2.$$

This is approximately equal to

$$\frac{1}{T_2 - T_1}[\langle B\rangle(T_2) - \langle B\rangle(T_1)] = \frac{T_2 - T_1}{T_2 - T_1} = 1.$$

As we increase the number of sample points, this approximation becomes exact. In other words, Brownian motion has *absolute volatility 1*.

Furthermore, consider the equation

$$\langle B\rangle(T) = T = \int_0^T 1\ dt, \qquad \forall T \ge 0.$$

This says that quadratic variation for Brownian motion accumulates at rate 1 *at all times along almost every path*.

## 14.5 Construction of the Itô Integral

The **integrator** is Brownian motion $B(t), t \geq 0$, with associated filtration $\mathcal{F}(t), t \geq 0$, and the following properties:

1. $s \leq t \Longrightarrow$ every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$,
2. $B(t)$ is $\mathcal{F}(t)$-measurable, $\forall t$,
3. For $t \leq t_1 \leq \ldots \leq t_n$, the increments $B(t_1) - B(t), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1})$ are independent of $\mathcal{F}(t)$.

The **integrand** is $\delta(t), t \geq 0$, where

1. $\delta(t)$ is $\mathcal{F}(t)$-measurable $\forall t$ (i.e., $\delta$ is adapted)
2. $\delta$ is square-integrable:

$$\mathbb{E} \int_0^T \delta^2(t)\, dt < \infty, \qquad \forall T.$$

We want to define the **Itô Integral:**

$$I(t) = \int_0^t \delta(u)\, dB(u), \qquad t \geq 0.$$

**Remark 14.3 (Integral w.r.t. a differentiable function)** If $f(t)$ is a differentiable function, then we can define

$$\int_0^t \delta(u)\, df(u) = \int_0^t \delta(u) f'(u)\, du.$$

This won't work when the integrator is Brownian motion, because the paths of Brownian motion are not differentiable.

## 14.6 Itô integral of an elementary integrand

Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0, T]$, i.e.,

$$0 = t_0 \leq t_1 \leq \ldots \leq t_n = T.$$

Assume that $\delta(t)$ is constant on each subinterval $[t_k, t_{k+1}]$ (see Fig. 14.2). We call such a $\delta$ an *elementary process*.

The functions $B(t)$ and $\delta(t_k)$ can be interpreted as follows:

- Think of $B(t)$ as the *price per unit share* of an asset at time $t$.

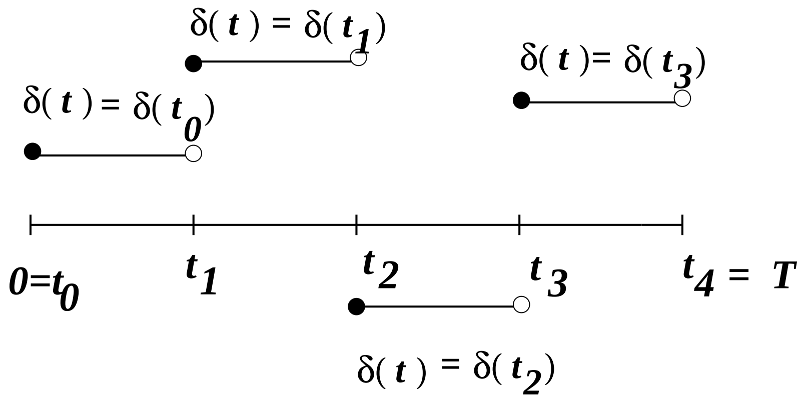


Figure 14.2: *An elementary function $\delta$.*

- Think of $t_0, t_1, \ldots, t_n$ as the *trading dates* for the asset.
- Think of $\delta(t_k)$ as the *number of shares of the asset acquired* at trading date $t_k$ and held until trading date $t_{k+1}$.

Then the Itô integral $I(t)$ can be interpreted as the *gain from trading* at time $t$; this gain is given by:

$$I(t) = \begin{cases} \delta(t_0)[B(t) - \underbrace{B(t_0)}_{=B(0)=0}], & 0 \le t \le t_1 \\ \delta(t_0)[B(t_1) - B(t_0)] + \delta(t_1)[B(t) - B(t_1)], & t_1 \le t \le t_2 \\ \delta(t_0)[B(t_1) - B(t_0)] + \delta(t_1)[B(t_2) - B(t_1)] + \delta(t_2)[B(t) - B(t_2)], & t_2 \le t \le t_3. \end{cases}$$

In general, if $t_k \le t \le t_{k+1}$,

$$I(t) = \sum_{j=0}^{k-1} \delta(t_j)[B(t_{j+1}) - B(t_j)] + \delta(t_k)[B(t) - B(t_k)].$$

## 14.7 Properties of the Itô integral of an elementary process

**Adaptedness** For each $t$, $I(t)$ is $\mathcal{F}(t)$-measurable.

**Linearity** If

$$I(t) = \int_0^t \delta(u)\, dB(u), \qquad J(t) = \int_0^t \gamma(u)\, dB(u)$$

then

$$I(t) \pm J(t) = \int_0^t (\delta(u) \pm \gamma(u))\, dB(u)$$

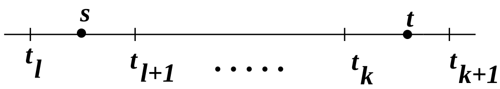


Figure 14.3: *Showing $s$ and $t$ in different partitions.*

and

$$cI(t) = \int_0^t c\delta(u)dB(u).$$

**Martingale** $I(t)$ is a martingale.

We prove the martingale property for the elementary process case.

**Theorem 7.45 (Martingale Property)**

$$I(t) = \sum_{j=0}^{k-1} \delta(t_j)[B(t_{j+1}) - B(t_j)] \; + \; \delta(t_k)[B(t) - B(t_k)], \qquad t_k \le t \le t_{k+1}$$

*is a martingale.*

**Proof:** Let $0 \le s \le t$ be given. We treat the more difficult case that $s$ and $t$ are in different subintervals, i.e., there are partition points $t_\ell$ and $t_k$ such that $s \in [t_\ell, t_{\ell+1}]$ and $t \in [t_k, t_{k+1}]$ (See Fig. 14.3).

Write

$$\begin{aligned} I(t) = \sum_{j=0}^{\ell-1} \delta(t_j)[B(t_{j+1}) - B(t_j)] \; + \; \delta(t_\ell)[B(t_{\ell+1}) - B(t_\ell)] \\ + \sum_{j=\ell+1}^{k-1} \delta(t_j)[B(t_{j+1}) - B(t_j)] \; + \; \delta(t_k)[B(t) - B(t_k)] \end{aligned}$$

We compute conditional expectations:

$$\begin{aligned} \mathbb{E}\left[\sum_{j=0}^{\ell-1} \delta(t_j)(B(t_{j+1}) - B(t_j)) \Big| \mathcal{F}(s)\right] &= \sum_{j=0}^{\ell-1} \delta(t_j)(B(t_{j+1}) - B(t_j)). \\ \mathbb{E}\left[\delta(t_\ell)(B(t_{\ell+1}) - B(t_\ell)) \Big| \mathcal{F}(s)\right] &= \delta(t_\ell)\left(\mathbb{E}[B(t_{\ell+1})|\mathcal{F}(s)] - B(t_\ell)\right) \\ &= \delta(t_\ell)[B(s) - B(t_\ell)] \end{aligned}$$

These first two terms add up to $I(s)$. We show that the third and fourth terms are zero.

$$\begin{aligned}
\mathbb{E}\left[\sum_{j=\ell+1}^{k-1}\delta(t_j)(B(t_{j+1})-B(t_j))\Big|\mathcal{F}(s)\right] &= \sum_{j=\ell+1}^{k-1}\mathbb{E}\left[\mathbb{E}\left[\delta(t_j)(B(t_{j+1})-B(t_j))\Big|\mathcal{F}(t_j)\right]\Big|\mathcal{F}(s)\right]\\
&= \sum_{j=\ell+1}^{k-1}\mathbb{E}\left[\delta(t_j)\underbrace{(\mathbb{E}[B(t_{j+1})|\mathcal{F}(t_j)]-B(t_j))}_{=0}\Big|\mathcal{F}(s)\right]
\end{aligned}$$

$$\mathbb{E}\left[\delta(t_k)(B(t)-B(t_k))\Big|\mathcal{F}(s)\right] = \mathbb{E}\left[\delta(t_k)\underbrace{(\mathbb{E}[B(t)|\mathcal{F}(t_k)]-B(t_k))}_{=0}\Big|\mathcal{F}(s)\right]$$

■

**Theorem 7.46 (Itô Isometry)**

$$\mathbb{E}I^2(t) = \mathbb{E}\int_0^t \delta^2(u)\,du.$$

**Proof:** To simplify notation, assume $t = t_k$, so

$$I(t) = \sum_{j=0}^{k}\delta(t_j)[\underbrace{B(t_{j+1})-B(t_j)}_{D_j}]$$

Each $D_j$ has expectation 0, and different $D_j$ are independent.

$$\begin{aligned}
I^2(t) &= \left(\sum_{j=0}^{k}\delta(t_j)D_j\right)^2\\
&= \sum_{j=0}^{k}\delta^2(t_j)D_j^2 + 2\sum_{i<j}\delta(t_i)\delta(t_j)D_iD_j.
\end{aligned}$$

Since the cross terms have expectation zero,

$$\begin{aligned}
\mathbb{E}I^2(t) &= \sum_{j=0}^{k}\mathbb{E}[\delta^2(t_j)D_j^2]\\
&= \sum_{j=0}^{k}\mathbb{E}\left[\delta^2(t_j)\mathbb{E}\left[(B(t_{j+1})-B(t_j))^2\Big|\mathcal{F}(t_j)\right]\right]\\
&= \sum_{j=0}^{k}\mathbb{E}\delta^2(t_j)(t_{j+1}-t_j)\\
&= \mathbb{E}\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}\delta^2(u)\,du\\
&= \mathbb{E}\int_0^t \delta^2(u)\,du
\end{aligned}$$

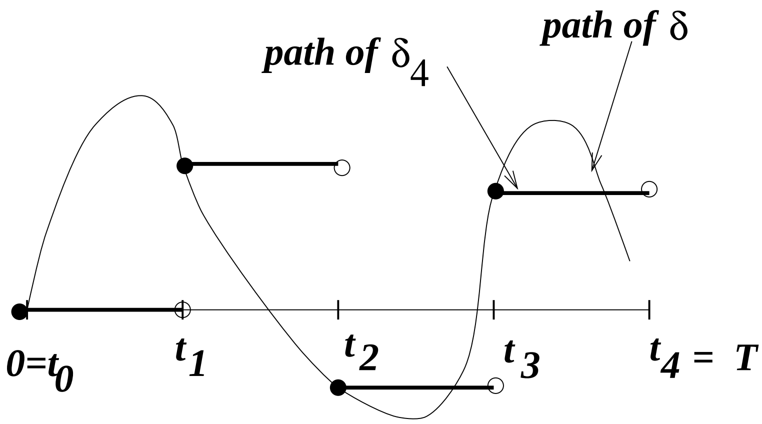


Figure 14.4: *Approximating a general process by an elementary process $\delta_4$, over $[0,T]$.*

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## 14.8 Itô integral of a general integrand

Fix $T > 0$. Let $\delta$ be a process (not necessarily an elementary process) such that

- $\delta(t)$ is $\mathcal{F}(t)$-measurable, $\forall t \in [0,T]$,
- $I\!E \int_0^T \delta^2(t)\ dt < \infty$.

**Theorem 8.47** *There is a sequence of elementary processes $\{\delta_n\}_{n=1}^{\infty}$ such that*

$$\lim_{n\to\infty} I\!E \int_0^T |\delta_n(t) - \delta(t)|^2\ dt = 0.$$

**Proof:** Fig. 14.4 shows the main idea.

■

In the last section we have defined

$$I_n(T) = \int_0^T \delta_n(t)\ dB(t)$$

for every $n$. We now define

$$\int_0^T \delta(t)\ dB(t) = \lim_{n\to\infty} \int_0^T \delta_n(t)\ dB(t).$$

The only difficulty with this approach is that we need to make sure the above limit exists. Suppose $n$ and $m$ are large positive integers. Then

$$\begin{aligned}
\mathrm{var}(I_n(T) - I_m(T)) &= \mathbb{E}\left(\int_0^T [\delta_n(t) - \delta_m(t)]\, dB(t)\right)^2 \\
\text{(Itô Isometry:)} &= \mathbb{E}\int_0^T [\delta_n(t) - \delta_m(t)]^2\, dt \\
&= \mathbb{E}\int_0^T [\, |\delta_n(t) - \delta(t)| + |\delta(t) - \delta_m(t)|\, ]^2\, dt \\
((a+b)^2 \le 2a^2 + 2b^2 :) &\le 2\mathbb{E}\int_0^T |\delta_n(t) - \delta(t)|^2\, dt + 2\mathbb{E}\int_0^T |\delta_m(t) - \delta(t)|^2\, dt,
\end{aligned}$$

which is small. This guarantees that the sequence $\{I_n(T)\}_{n=1}^{\infty}$ has a limit.

## 14.9 Properties of the (general) Itô integral

$$I(t) = \int_0^t \delta(u)\, dB(u).$$

Here $\delta$ is any adapted, square-integrable process.

**Adaptedness.** For each $t$, $I(t)$ is $\mathcal{F}(t)$-measurable.

**Linearity.** If

$$I(t) = \int_0^t \delta(u)\, dB(u), \qquad J(t) = \int_0^t \gamma(u)\, dB(u)$$

then

$$I(t) \pm J(t) = \int_0^t (\delta(u) \pm \gamma(u))\, dB(u)$$

and

$$cI(t) = \int_0^t c\delta(u) dB(u).$$

**Martingale.** $I(t)$ is a martingale.

**Continuity.** $I(t)$ is a continuous function of the upper limit of integration $t$.

**Itô Isometry.** $\mathbb{E} I^2(t) = \mathbb{E}\int_0^t \delta^2(u)\, du$.

**Example 14.1 ()** Consider the Itô integral

$$\int_0^T B(u)\, dB(u).$$

We approximate the integrand as shown in Fig. 14.5

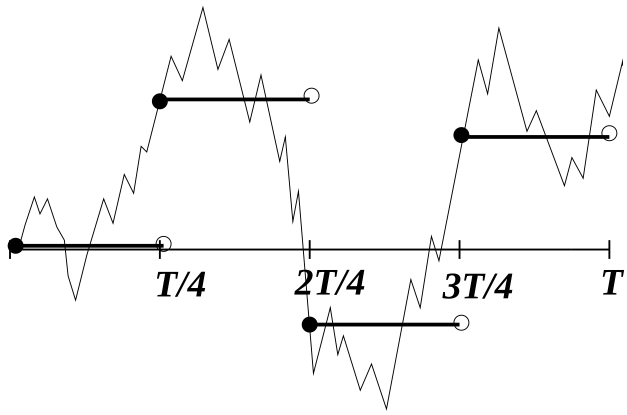


Figure 14.5: *Approximating the integrand* $B(u)$ *with* $\delta_4$, *over* $[0, T]$.

$$\delta_n(u) = \begin{cases} B(0) = 0 & \text{if} \quad 0 \le u < T/n; \\ B(T/n) & \text{if} \quad T/n \le u < 2T/n; \\ \dots & \\ B\left(\frac{(n-1)T}{T}\right) & \text{if} \quad \frac{(n-1)T}{n} \le u < T. \end{cases}$$

By definition,

$$\int_0^T B(u)\; dB(u) = \lim_{n\to\infty} \sum_{k=0}^{n-1} B\left(\frac{kT}{n}\right)\left[B\left(\frac{(k+1)T}{n}\right) - B\left(\frac{kT}{n}\right)\right].$$

To simplify notation, we denote

$$B_k \triangleq B\left(\frac{kT}{n}\right),$$

so

$$\int_0^T B(u)\; dB(u) = \lim_{n\to\infty} \sum_{k=0}^{n-1} B_k(B_{k+1} - B_k).$$

We compute

$$\begin{aligned}
\tfrac{1}{2}\sum_{k=0}^{n-1}(B_{k+1} - B_k)^2 &= \tfrac{1}{2}\sum_{k=0}^{n-1} B_{k+1}^2 - \sum_{k=0}^{n-1} B_k B_{k+1} + \tfrac{1}{2}\sum_{k=0}^{n-1} B_k^2 \\
&= \tfrac{1}{2}B_n^2 + \tfrac{1}{2}\sum_{j=0}^{n-1} B_j^2 - \sum_{k=0}^{n-1} B_k B_{k+1} + \tfrac{1}{2}\sum_{k=0}^{n-1} B_k^2 \\
&= \tfrac{1}{2}B_n^2 + \sum_{k=0}^{n-1} B_k^2 - \sum_{k=0}^{n-1} B_k B_{k+1} \\
&= \tfrac{1}{2}B_n^2 - \sum_{k=0}^{n-1} B_k(B_{k+1} - B_k).
\end{aligned}$$

Therefore,

$$\sum_{k=0}^{n-1} B_k(B_{k+1} - B_k) = \tfrac{1}{2}B_n^2 - \tfrac{1}{2}\sum_{k=0}^{n-1}(B_{k+1} - B_k)^2,$$

or equivalently

$$\sum_{k=0}^{n-1} B\left(\frac{kT}{n}\right)\left[B\left(\frac{(k+1)T}{n}\right) - B\left(\frac{kT}{n}\right)\right] = \tfrac{1}{2}B^2(T) - \tfrac{1}{2}\sum_{k=0}^{n-1}\left[B\left(\frac{(k+1)T}{n}\right)\left(\frac{k}{T}\right)\right]^2.$$

Let $n\to\infty$ and use the definition of quadratic variation to get

$$\int_0^T B(u)\, dB(u) = \tfrac{1}{2}B^2(T) - \tfrac{1}{2}T.$$

■

**Remark 14.4 (Reason for the $\frac{1}{2}T$ term)** If $f$ is differentiable with $f(0) = 0$, then

$$\begin{aligned}\int_0^T f(u)\, df(u) &= \int_0^T f(u)f'(u)\, du \\ &= \tfrac{1}{2}f^2(u)\Big|_0^T \\ &= \tfrac{1}{2}f^2(T).\end{aligned}$$

In contrast, for Brownian motion, we have

$$\int_0^T B(u)dB(u) = \tfrac{1}{2}B^2(T) - \tfrac{1}{2}T.$$

The extra term $\frac{1}{2}T$ comes from the nonzero quadratic variation of Brownian motion. It has to be there, because

$$\mathbb{E}\int_0^T B(u)\, dB(u) = 0 \qquad \text{(Itô integral is a martingale)}$$

but

$$\mathbb{E}\tfrac{1}{2}B^2(T) = \tfrac{1}{2}T.$$

## 14.10 Quadratic variation of an Itô integral

**Theorem 10.48 (Quadratic variation of Itô integral)** *Let*

$$I(t) = \int_0^t \delta(u)\, dB(u).$$

*Then*

$$\langle I\rangle(t) = \int_0^t \delta^2(u)\, du.$$

This holds even if $\delta$ is not an elementary process. The quadratic variation formula says that at each time $u$, the *instantaneous absolute volatility* of $I$ is $\delta^2(u)$. This is the absolute volatility of the Brownian motion scaled by the size of the position (i.e. $\delta(t)$) in the Brownian motion. Informally, we can write the quadratic variation formula in differential form as follows:

$$dI(t)\ dI(t) = \delta^2(t)\ dt.$$

Compare this with

$$dB(t)\ dB(t) = dt.$$

**Proof:** (For an elementary process $\delta$). Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be the partition for $\delta$, i.e., $\delta(t) = \delta(t_k)$ for $t_k \le t \le t_{k+1}$. To simplify notation, assume $t = t_n$. We have

$$\langle I\rangle(t) = \sum_{k=0}^{n-1} \left[\langle I\rangle(t_{k+1}) - \langle I\rangle(t_k)\right].$$

Let us compute $\langle I\rangle(t_{k+1}) - \langle I\rangle(t_k)$. Let $\Xi = \{s_0, s_1, \ldots, s_m\}$ be a partition

$$t_k = s_0 \le s_1 \le \ldots \le s_m = t_{k+1}.$$

Then

$$\begin{aligned} I(s_{j+1}) - I(s_j) &= \int_{s_j}^{s_{j+1}} \delta(t_k)\ dB(u) \\ &= \delta(t_k)\left[B(s_{j+1}) - B(s_j)\right], \end{aligned}$$

so

$$\begin{aligned} \langle I\rangle(t_{k+1}) - \langle I\rangle(t_k) &= \sum_{j=0}^{m-1} \left[I(s_{j+1}) - I(s_j)\right]^2 \\ &= \delta^2(t_k) \sum_{j=0}^{m-1} \left[B(s_{j+1}) - B(s_j)\right]^2 \\ &\xrightarrow{||\Xi||\to 0} \delta^2(t_k)(t_{k+1} - t_k). \end{aligned}$$

It follows that

$$\begin{aligned} \langle I\rangle(t) &= \sum_{k=0}^{n-1} \delta^2(t_k)(t_{k+1} - t_k) \\ &= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \delta^2(u)\ du \\ &\xrightarrow{||\Pi||\to 0} \int_0^t \delta^2(u)\ du. \end{aligned}$$

■

# Chapter 15

# Itô's Formula

## 15.1 Itô's formula for one Brownian motion

We want a rule to "differentiate" expressions of the form $f(B(t))$, where $f(x)$ is a differentiable function. If $B(t)$ were also differentiable, then the ordinary *chain rule* would give

$$\frac{d}{dt} f(B(t)) = f'(B(t))B'(t),$$

which could be written in differential notation as

$$\begin{aligned} df(B(t)) &= f'(B(t))B'(t)\ dt \\ &= f'(B(t))dB(t) \end{aligned}$$

However, $B(t)$ is not differentiable, and in particular has nonzero quadratic variation, so the correct formula has an extra term, namely,

$$df(B(t)) = f'(B(t))\ dB(t) + \tfrac{1}{2} f''(B(t)) \underbrace{dt}_{dB(t)\ dB(t)}.$$

This is *Itô's formula in differential form.* Integrating this, we obtain *Itô's formula in integral form:*

$$f(B(t)) - \underbrace{f(B(0))}_{f(0)} = \int_0^t f'(B(u))\ dB(u) + \tfrac{1}{2} \int_0^t f''(B(u))\ du.$$

**Remark 15.1 (Differential vs. Integral Forms)** The mathematically meaningful form of Itô's formula is Itô's formula in integral form:

$$f(B(t)) - f(B(0)) = \int_0^t f'(B(u))\ dB(u) + \tfrac{1}{2} \int_0^t f''(B(u))\ du.$$

This is because we have solid definitions for both integrals appearing on the right-hand side. The first,

$$\int_0^t f'(B(u))\,dB(u)$$

is an *Itô integral*, defined in the previous chapter. The second,

$$\int_0^t f''(B(u))\,du,$$

is a *Riemann integral*, the type used in freshman calculus.

For paper and pencil computations, the more convenient form of Itô's rule is *Itô's formula in differential form:*

$$df(B(t)) = f'(B(t))\,dB(t) + \tfrac{1}{2}f''(B(t))\,dt.$$

There is an intuitive meaning but no solid definition for the terms $df(B(t))$, $dB(t)$ and $dt$ appearing in this formula. This formula becomes mathematically respectable only after we integrate it.

## 15.2 Derivation of Itô's formula

Consider $f(x) = \frac{1}{2}x^2$, so that

$$f'(x) = x, \quad f''(x) = 1.$$

Let $x_k, x_{k+1}$ be numbers. Taylor's formula implies

$$f(x_{k+1}) - f(x_k) = (x_{k+1} - x_k)f'(x_k) + \tfrac{1}{2}(x_{k+1} - x_k)^2 f''(x_k).$$

In this case, Taylor's formula to second order is *exact* because $f$ is a *quadratic function*.

In the general case, the above equation is only approximate, and the error is of the order of $(x_{k+1} - x_k)^3$. The total error will have limit zero in the last step of the following argument.

Fix $T > 0$ and let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0, T]$. Using Taylor's formula, we write:

$$\begin{aligned}
&f(B(T)) - f(B(0)) \\
&= \tfrac{1}{2}B^2(T) - \tfrac{1}{2}B^2(0) \\
&= \sum_{k=0}^{n-1} \left[f(B(t_{k+1})) - f(B(t_k))\right] \\
&= \sum_{k=0}^{n-1} \left[B(t_{k+1}) - B(t_k)\right] f'(B(t_k)) + \tfrac{1}{2}\sum_{k=0}^{n-1} \left[B(t_{k+1}) - B(t_k)\right]^2 f''(B(t_k)) \\
&= \sum_{k=0}^{n-1} B(t_k)\left[B(t_{k+1}) - B(t_k)\right] + \tfrac{1}{2}\sum_{k=0}^{n-1} \left[B(t_{k+1}) - B(t_k)\right]^2.
\end{aligned}$$

We let $||\Pi||\to 0$ to obtain

$$\begin{aligned} f(B(T)) - f(B(0)) &= \int_0^T B(u)\ dB(u) + \tfrac{1}{2} \underbrace{\langle B\rangle(T)}_{T} \\ &= \int_0^T f'(B(u))\ dB(u) + \tfrac{1}{2}\int_0^T \underbrace{f''(B(u))}_{1}\, du. \end{aligned}$$

This is Itô's formula in integral form for the special case

$$f(x) = \tfrac{1}{2}x^2.$$

## 15.3 Geometric Brownian motion

**Definition 15.1 (Geometric Brownian Motion)** Geometric Brownian motion is

$$S(t) = S(0)\exp\left\{\sigma B(t) + \left(\mu - \tfrac{1}{2}\sigma^2\right)t\right\},$$

where $\mu$ and $\sigma > 0$ are constant.

Define

$$f(t,x) = S(0)\exp\left\{\sigma x + \left(\mu - \tfrac{1}{2}\sigma^2\right)t\right\},$$

so

$$S(t) = f(t, B(t)).$$

Then

$$f_t = \left(\mu - \tfrac{1}{2}\sigma^2\right)f,\ f_x = \sigma f,\ f_{xx} = \sigma^2 f.$$

According to Itô's formula,

$$\begin{aligned} dS(t) &= df(t, B(t)) \\ &= f_t dt + f_x dB + \tfrac{1}{2} f_{xx} \underbrace{dBdB}_{dt} \\ &= (\mu - \tfrac{1}{2}\sigma^2) f\ dt + \sigma f\ dB + \tfrac{1}{2}\sigma^2 f\ dt \\ &= \mu S(t) dt + \sigma S(t)\ dB(t) \end{aligned}$$

Thus, *Geometric Brownian motion in differential form* is

$$dS(t) = \mu S(t) dt + \sigma S(t)\ dB(t),$$

and *Geometric Brownian motion in integral form* is

$$S(t) = S(0) + \int_0^t \mu S(u)\ du + \int_0^t \sigma S(u)\ dB(u).$$

## 15.4 Quadratic variation of geometric Brownian motion

In the integral form of Geometric Brownian motion,

$$S(t) = S(0) + \int_0^t \mu S(u)\,du + \int_0^t \sigma S(u)\,dB(u),$$

the Riemann integral

$$F(t) = \int_0^t \mu S(u)\,du$$

is differentiable with $F'(t) = \mu S(t)$. This term has zero quadratic variation. The Itô integral

$$G(t) = \int_0^t \sigma S(u)\,dB(u)$$

is not differentiable. It has quadratic variation

$$\langle G\rangle(t) = \int_0^t \sigma^2 S^2(u)\,du.$$

Thus the quadratic variation of $S$ is given by the quadratic variation of $G$. In differential notation, we write

$$dS(t)\,dS(t) = (\mu S(t)dt + \sigma S(t)dB(t))^2 = \sigma^2 S^2(t)\,dt$$

## 15.5 Volatility of Geometric Brownian motion

Fix $0 \leq T_1 \leq T_2$. Let $\Pi = \{t_0, \ldots, t_n\}$ be a partition of $[T_1, T_2]$. The *squared absolute sample volatility* of $S$ on $[T_1, T_2]$ is

$$\begin{aligned} \frac{1}{T_2 - T_1}\sum_{k=0}^{n-1}[S(t_{k+1}) - S(t_k)]^2 &\simeq \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} \sigma^2 S^2(u)\,du \\ &\simeq \sigma^2 S^2(T_1) \end{aligned}$$

As $T_2 \downarrow T_1$, the above approximation becomes exact. In other words, the *instantaneous relative volatility* of $S$ is $\sigma^2$. This is usually called simply the *volatility* of $S$.

## 15.6 First derivation of the Black-Scholes formula

**Wealth of an investor.** An investor begins with nonrandom initial wealth $X_0$ and at each time $t$, holds $\Delta(t)$ shares of stock. Stock is modelled by a geometric Brownian motion:

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t).$$

$\Delta(t)$ can be random, but must be adapted. The investor finances his investing by borrowing or lending at interest rate $r$.

Let $X(t)$ denote the wealth of the investor at time $t$. Then

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + r\left[X(t) - \Delta(t)S(t)\right]dt \\ &= \Delta(t)\left[\mu S(t)dt + \sigma S(t)dB(t)\right] + r\left[X(t) - \Delta(t)S(t)\right]dt \\ &= rX(t)dt \;+\; \Delta(t)S(t)\underbrace{(\mu - r)}_{\text{Risk premium}}\,dt \;+\; \Delta(t)S(t)\sigma dB(t). \end{aligned}$$

**Value of an option.** Consider an European option which pays $g(S(T))$ at time $T$. Let $v(t,x)$ denote the value of this option at time $t$ if the stock price is $S(t) = x$. In other words, the value of the option at each time $t \in [0,T]$ is

$$v(t, S(t)).$$

The differential of this value is

$$\begin{aligned} dv(t,S(t)) &= v_t dt + v_x dS + \tfrac{1}{2}v_{xx}dS\;dS \\ &= v_t dt + v_x\left[\mu S\;dt + \sigma S\;dB\right] + \tfrac{1}{2}v_{xx}\sigma^2 S^2\;dt \\ &= \left[v_t + \mu S v_x + \tfrac{1}{2}\sigma^2 S^2 v_{xx}\right]\;dt + \sigma S v_x dB \end{aligned}$$

A hedging portfolio starts with some initial wealth $X_0$ and invests so that the wealth $X(t)$ at each time tracks $v(t,S(t))$. We saw above that

$$dX(t) = \left[rX + \Delta(\mu - r)S\right]\;dt + \sigma S\Delta dB.$$

To ensure that $X(t) = v(t,S(t))$ for all $t$, we equate coefficients in their differentials. Equating the $dB$ coefficients, we obtain the $\Delta$-*hedging rule*:

$$\Delta(t) = v_x(t, S(t)).$$

Equating the $dt$ coefficients, we obtain:

$$v_t + \mu S v_x + \tfrac{1}{2}\sigma^2 S^2 v_{xx} = rX + \Delta(\mu - r)S.$$

But we have set $\Delta = v_x$, and we are seeking to cause $X$ to agree with $v$. Making these substitutions, we obtain

$$v_t + \mu S v_x + \tfrac{1}{2}\sigma^2 S^2 v_{xx} = rv + v_x(\mu - r)S,$$

(where $v = v(t,S(t))$ and $S = S(t)$) which simplifies to

$$v_t + rSv_x + \tfrac{1}{2}\sigma^2 S^2 v_{xx} = rv.$$

In conclusion, we should let $v$ be the solution to the *Black-Scholes partial differential equation*

$$v_t(t,x) + rxv_x(t,x) + \tfrac{1}{2}\sigma^2 x^2 v_{xx}(t,x) = rv(t,x)$$

satisfying the terminal condition

$$v(T,x) = g(x).$$

If an investor starts with $X_0 = v(0,S(0))$ and uses the hedge $\Delta(t) = v_x(t,S(t))$, then he will have $X(t) = v(t,S(t))$ for all $t$, and in particular, $X(T) = g(S(T))$.

## 15.7 Mean and variance of the Cox-Ingersoll-Ross process

The *Cox-Ingersoll-Ross* model for interest rates is

$$dr(t) = a(b - cr(t))dt + \sigma\sqrt{r(t)}\; dB(t),$$

where $a, b, c, \sigma$ and $r(0)$ are positive constants. In integral form, this equation is

$$r(t) = r(0) + a\int_0^t (b - cr(u))\; du + \sigma\int_0^t \sqrt{r(u)}\; dB(u).$$

We apply Itô's formula to compute $dr^2(t)$. This is $df(r(t))$, where $f(x) = x^2$. We obtain

$$\begin{aligned} dr^2(t) &= df(r(t)) \\ &= f'(r(t))\; dr(t) + \tfrac{1}{2}f''(r(t))\; dr(t)\; dr(t) \\ &= 2r(t)\left[a(b - cr(t))\; dt + \sigma\sqrt{r(t)}\; dB(t)\right] + \left[a(b - cr(t))\; dt + \sigma\sqrt{r(t)}\; dB(t)\right]^2 \\ &= 2abr(t)\; dt - 2acr^2(t)\; dt + 2\sigma r^{\frac{3}{2}}(t)\; dB(t) + \sigma^2 r(t)\; dt \\ &= (2ab + \sigma^2)r(t)\; dt - 2acr^2(t)\; dt + 2\sigma r^{\frac{3}{2}}(t)\; dB(t) \end{aligned}$$

**The mean of $r(t)$.** The integral form of the CIR equation is

$$r(t) = r(0) + a\int_0^t (b - cr(u))\; du + \sigma\int_0^t \sqrt{r(u)}\; dB(u).$$

Taking expectations and remembering that the expectation of an Itô integral is zero, we obtain

$$I\!\!E r(t) = r(0) + a\int_0^t (b - cI\!\!E r(u))\; du.$$

Differentiation yields

$$\frac{d}{dt}I\!\!E r(t) = a(b - cI\!\!E r(t)) = ab - acI\!\!E r(t),$$

which implies that

$$\frac{d}{dt}\left[e^{act}I\!\!E r(t)\right] = e^{act}\left[acI\!\!E r(t) + \frac{d}{dt}I\!\!E r(t)\right] = e^{act}ab.$$

Integration yields

$$e^{act}I\!\!E r(t) - r(0) = ab\int_0^t e^{acu}\; du = \frac{b}{c}(e^{act} - 1).$$

We solve for $I\!\!E r(t)$:

$$I\!\!E r(t) = \frac{b}{c} + e^{-act}\left(r(0) - \frac{b}{c}\right).$$

If $r(0) = \frac{b}{c}$, then $I\!\!E r(t) = \frac{b}{c}$ for every $t$. If $r(0) \neq \frac{b}{c}$, then $r(t)$ exhibits *mean reversion*:

$$\lim_{t\to\infty} I\!\!E r(t) = \frac{b}{c}.$$

**Variance of** $r(t)$. The integral form of the equation derived earlier for $dr^2(t)$ is

$$r^2(t) = r^2(0) + (2ab + \sigma^2) \int_0^t r(u)\, du - 2ac \int_0^t r^2(u)\, du + 2\sigma \int_0^t r^{\frac{3}{2}}(u)\, dB(u).$$

Taking expectations, we obtain

$$\mathbb{E}r^2(t) = r^2(0) + (2ab + \sigma^2) \int_0^t \mathbb{E}r(u)\, du - 2ac \int_0^t \mathbb{E}r^2(u)\, du.$$

Differentiation yields

$$\frac{d}{dt}\mathbb{E}r^2(t) = (2ab + \sigma^2)\mathbb{E}r(t) - 2ac\mathbb{E}r^2(t),$$

which implies that

$$\begin{aligned}\frac{d}{dt}e^{2act}\mathbb{E}r^2(t) &= e^{2act}\left[2ac\mathbb{E}r^2(t) + \frac{d}{dt}\mathbb{E}r^2(t)\right]\\ &= e^{2act}(2ab + \sigma^2)\mathbb{E}r(t).\end{aligned}$$

Using the formula already derived for $\mathbb{E}r(t)$ and integrating the last equation, after considerable algebra we obtain

$$\begin{aligned}\mathbb{E}r^2(t) &= \frac{b\sigma^2}{2ac^2} + \frac{b^2}{c^2} + \left(r(0) - \frac{b}{c}\right)\left(\frac{\sigma^2}{ac} + \frac{2b}{c}\right)e^{-act}\\ &\quad + \left(r(0) - \frac{b}{c}\right)^2 \frac{\sigma^2}{ac}e^{-2act} + \frac{\sigma^2}{ac}\left(\frac{b}{2c} - r(0)\right)e^{-2act}.\\ \operatorname{var} r(t) &= \mathbb{E}r^2(t) - (\mathbb{E}r(t))^2\\ &= \frac{b\sigma^2}{2ac^2} + \left(r(0) - \frac{b}{c}\right)\frac{\sigma^2}{ac}e^{-act} + \frac{\sigma^2}{ac}\left(\frac{b}{2c} - r(0)\right)e^{-2act}.\end{aligned}$$

## 15.8 Multidimensional Brownian Motion

**Definition 15.2 ($d$-dimensional Brownian Motion)** A *$d$-dimensional Brownian Motion* is a process

$$B(t) = (B_1(t), \ldots, B_d(t))$$

with the following properties:

- Each $B_k(t)$ is a one-dimensional Brownian motion;
- If $i \neq j$, then the processes $B_i(t)$ and $B_j(t)$ are independent.

Associated with a $d$-dimensional Brownian motion, we have a filtration $\{\mathcal{F}(t)\}$ such that

- For each $t$, the random vector $B(t)$ is $\mathcal{F}(t)$-measurable;
- For each $t \leq t_1 \leq \ldots \leq t_n$, the vector increments

  $$B(t_1) - B(t), \ldots, B(t_n) - B(t_{n-1})$$

  are independent of $\mathcal{F}(t)$.

## 15.9 Cross-variations of Brownian motions

Because each component $B_i$ is a one-dimensional Brownian motion, we have the informal equation

$$dB_i(t)\ dB_i(t) = dt.$$

However, we have:

**Theorem 9.49** *If $i \neq j$,*

$$dB_i(t)\ dB_j(t) = 0$$

**Proof:** Let $\Pi = \{t_0, \ldots, t_n\}$ be a partition of $[0,T]$. For $i \neq j$, define the *sample cross variation* of $B_i$ and $B_j$ on $[0,T]$ to be

$$C_\Pi = \sum_{k=0}^{n-1} \left[B_i(t_{k+1}) - B_i(t_k)\right]\left[B_j(t_{k+1}) - B_j(t_k)\right].$$

The increments appearing on the right-hand side of the above equation are all independent of one another and all have mean zero. Therefore,

$$I\!\!E C_\Pi = 0.$$

We compute $\text{var}(C_\Pi)$. First note that

$$\begin{aligned} C_\Pi^2 &= \sum_{k=0}^{n-1} \Big[B_i(t_{k+1}) - B_i(t_k)\Big]^2 \Big[B_j(t_{k+1}) - B_j(t_k)\Big]^2 \\ &+ 2\sum_{\ell<k}^{n-1} \left[B_i(t_{\ell+1}) - B_i(t_\ell)\right]\left[B_j(t_{\ell+1}) - B_j(t_\ell)\right].\left[B_i(t_{k+1}) - B_i(t_k)\right]\left[B_j(t_{k+1}) - B_j(t_k)\right] \end{aligned}$$

All the increments appearing in the sum of cross terms are independent of one another and have mean zero. Therefore,

$$\begin{aligned} \text{var}(C_\Pi) &= I\!\!E C_\Pi^2 \\ &= I\!\!E \sum_{k=0}^{n-1} \left[B_i(t_{k+1}) - B_i(t_k)\right]^2 \left[B_j(t_{k+1}) - B_j(t_k)\right]^2. \end{aligned}$$

But $[B_i(t_{k+1}) - B_i(t_k)]^2$ and $[B_j(t_{k+1}) - B_j(t_k)]^2$ are independent of one another, and each has expectation $(t_{k+1} - t_k)$. It follows that

$$\text{var}(C_\Pi) = \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2 \leq ||\Pi|| \sum_{k=0}^{n-1} (t_{k+1} - t_k) = ||\Pi||.T.$$

As $||\Pi|| \to 0$, we have $\text{var}(C_\Pi) \to 0$, so $C_\Pi$ converges to the constant $I\!\!E C_\Pi = 0$. ∎

## 15.10 Multi-dimensional Itô formula

To keep the notation as simple as possible, we write the Itô formula for *two* processes driven by a *two*-dimensional Brownian motion. The formula generalizes to *any number* of processes driven by a Brownian motion of *any number* (not necessarily the same number) of dimensions.

Let $X$ and $Y$ be processes of the form

$$X(t) = X(0) + \int_0^t \alpha(u)\, du + \int_0^t \delta_{11}(u)\, dB_1(u) + \int_0^t \delta_{12}(u)\, dB_2(u),$$
$$Y(t) = Y(0) + \int_0^t \beta(u)\, du + \int_0^t \delta_{21}(u)\, dB_1(u) + \int_0^t \delta_{22}(u)\, dB_2(u).$$

Such processes, consisting of a nonrandom initial condition, plus a Riemann integral, plus one or more Itô integrals, are called *semimartingales*. The integrands $\alpha(u), \beta(u)$, and $\delta_{ij}(u)$ can be any adapted processes. The adaptedness of the integrands guarantees that $X$ and $Y$ are also adapted. In differential notation, we write

$$dX = \alpha\, dt + \delta_{11}\, dB_1 + \delta_{12}\, dB_2,$$
$$dY = \beta\, dt + \delta_{21}\, dB_1 + \delta_{22}\, dB_2.$$

Given these two semimartingales $X$ and $Y$, the quadratic and cross variations are:

$$\begin{aligned}
dX\, dX &= (\alpha\, dt + \delta_{11}\, dB_1 + \delta_{12}\, dB_2)^2, \\
&= \delta_{11}^2 \underbrace{dB_1\, dB_1}_{dt} + 2\delta_{11}\delta_{12} \underbrace{dB_1\, dB_2}_{0} + \delta_{12}^2 \underbrace{dB_2\, dB_2}_{dt} \\
&= (\delta_{11}^2 + \delta_{12}^2)^2\, dt, \\
dY\, dY &= (\beta\, dt + \delta_{21}\, dB_1 + \delta_{22}\, dB_2)^2 \\
&= (\delta_{21}^2 + \delta_{22}^2)^2\, dt, \\
dX\, dY &= (\alpha\, dt + \delta_{11}\, dB_1 + \delta_{12}\, dB_2)(\beta\, dt + \delta_{21}\, dB_1 + \delta_{22}\, dB_2) \\
&= (\delta_{11}\delta_{21} + \delta_{12}\delta_{22})\, dt
\end{aligned}$$

Let $f(t, x, y)$ be a function of three variables, and let $X(t)$ and $Y(t)$ be semimartingales. Then we have the corresponding Itô formula:

$$df(t, x, y) = f_t\, dt + f_x\, dX + f_y\, dY + \tfrac{1}{2}\left[f_{xx}\, dX\, dX + 2f_{xy}\, dX\, dY + f_{yy}\, dY\, dY\right].$$

In integral form, with $X$ and $Y$ as decribed earlier and with all the variables filled in, this equation is

$$\begin{aligned}
&f(t, X(t), Y(t)) - f(0, X(0), Y(0)) \\
&= \int_0^t [f_t + \alpha f_x + \beta f_y + \tfrac{1}{2}(\delta_{11}^2 + \delta_{12}^2) f_{xx} + (\delta_{11}\delta_{21} + \delta_{12}\delta_{22}) f_{xy} + \tfrac{1}{2}(\delta_{21}^2 + \delta_{22}^2) f_{yy}]\, du \\
&\quad + \int_0^t [\delta_{11} f_x + \delta_{21} f_y]\, dB_1 + \int_0^t [\delta_{12} f_x + \delta_{22} f_y]\, dB_2,
\end{aligned}$$

where $f = f(u, X(u), Y(u))$, for $i, j \in \{1, 2\}$, $\delta_{ij} = \delta_{ij}(u)$, and $B_i = B_i(u)$.