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NOTES AND COMMENTS

POVERTY ORDERINGS

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1. INTRODUCTION

THE PROBLEM OF DETERMINING an appropriate poverty line, and thus identifying those who are classified as poor, has always been one of the principal methodological issues in the analysis of poverty. Various procedures have been developed, based on alternative concepts of poverty. But a feature common to all proposed methods is a significant degree of arbitrariness in the value assigned to the poverty standard.[2] This is evident even in approaches based on subsistence needs since "there is no one level of food intake required for subsistence, but rather a broad range where physical efficiency declines with a falling intake of calories and proteins" (Atkinson (1983), p. 226).

Recent years have witnessed a shift of emphasis in the discussion of methodological issues underlying poverty analysis, away from the *identification* problem concerned with setting the poverty standard, and towards the *aggregation* problem of constructing a suitable overall index of poverty. Following Sen (1976, 1979), the widely used headcount and poverty gap type measures have been supplemented by a host of "distribution sensitive" indices, each of which is justified in terms of its particular properties: see, for example, Anand (1977, 1983), Thon (1979), Blackorby and Donaldson (1980), and Foster et al. (1984).[3] The value assigned to the poverty line is a critical parameter in these poverty indices. But in order to focus attention on the aggregation issue, the poverty line is always assumed to be fixed and given in advance. Thus the derivation of a poverty index and its implied poverty ordering of distributions has been treated as a topic quite separate from that of setting the poverty standard.

This clear cut distinction between the identification and aggregation aspects of poverty measurement may be less easy to maintain in practice, when attempting to decide which of two distributions exhibits more poverty. It is true that a poverty index will rank any pair of distributions once the poverty line is chosen. But any evidence that points to, say, distribution F having less poverty than distribution G would have little significance if the ranking is reversed at some reasonable alternative poverty line. In these circumstances we may simply have to conclude that the poverty comparison is ambiguous or inconclusive. On the other hand, if *no* reversals of the original ranking are found at *any* other potential poverty line, we can claim with confidence that F has *unambiguously* less poverty than G relative to the poverty index and range of poverty lines under consideration.

This paper will explore the partial ordering of distributions induced by such unambiguous poverty judgements, to provide a new perspective on poverty measurement and its relation to inequality and welfare. Our principal result characterizes the unambiguous comparisons associated with the class of measures P_α proposed by Foster et al. (1984), when the poverty standard may take any positive value. If α is a positive integer we find that the poverty orderings correspond precisely to the α-degree stochastic dominance partial orderings. Interesting welfare interpretations of the poverty orderings can be given for three members of the class P_α. Furthermore, we show that the ordering associated with P_2 (the per capita income gap) corresponds to the generalized Lorenz

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[2] For descriptions of the conceptual and practical difficulties, see Atkinson (1983) or Townsend (1979).

[3] See also the surveys of Sen (1979, 1981) and Foster (1984).

criterion discussed in Shorrocks (1983), and degenerates to the Lorenz inequality ranking for distributions with a common mean.

2. DEFINITIONS

We consider income distributions represented by distribution functions drawn from the set

$$\mathscr{F} := \{ F: \mathbb{R}_+ \to [0,1] \mid F \text{ is nondecreasing and right continuous}; \quad F(0) = 0; \text{ and } F(s_F) = 1 \text{ for some } s_F < \infty \}^4$$

where $\mathbb{R}_+ := [0, \infty)$. The mean of $F \in \mathscr{F}$ is denoted by $\mu_F := \int_0^\infty s\, dF(s)$ and the generalized inverse F^{-1} is defined as $F^{-1}(p) := \inf\{ s \geqslant 0 \mid F(s) \geqslant p\}$ for $p \in [0,1]$.

A poverty index is a function $P: \mathscr{F} \times \mathbb{R}_{++} \to \mathbb{R}$ whose value $P(F; z)$ indicates the degree of poverty associated with distribution F when the poverty line is z. One example is the *headcount ratio*

$$(1) \qquad P_1(F; z) := F(z),$$

corresponding to the proportion of the population lying at or below the poverty line. Another is the *per capita income gap*

$$(2) \qquad P_2(F; z) := \frac{1}{z} \int_0^{F(z)} [z - F^{-1}(p)]\, dp,$$

which is a normalized sum of the shortfalls of the poor. Sen (1976) and others have argued for indices that are also sensitive to the distribution of income among the poor. One index of this type, recommended by Foster et al. (1984), can be written as

$$(3) \qquad P_3(F; z) := \frac{1}{z^2} \int_0^{F(z)} [z - F^{-1}(p)]^2\, dp.$$

This is simply a weighted sum of shortfalls of the poor, where the weights are the shortfalls themselves. Thus, it attaches greater weight to lower incomes amongst the poor. In fact, Foster et al. (1984) propose a general class of poverty indices given by

$$(4) \qquad P_\alpha(F; z) := \frac{1}{z^{\alpha-1}} \int_0^{F(z)} [z - F^{-1}(p)]^{\alpha-1}\, dp, \quad \alpha \geqslant 1,$$

which includes as special cases not only the distribution sensitive index P_3 above, but also the headcount ratio P_1 and the per capita income gap P_2.

3. POVERTY ORDERINGS

The uncertainty which surrounds the appropriate level of the poverty standard suggests that we should postulate a range Z of reasonable poverty lines, rather than a particular fixed value z. This raises the possibility of ambiguous poverty comparisons. For we may

[4] The assumption that each $F \in \mathscr{F}$ has a finite maximum income is not needed for Proposition 1, but it helps with the welfare interpretation provided by Proposition 2.

have $P(F; z_0) > P(G; z_0)$ when the poverty line is set at $z_0 \in Z$, and the opposite result $P(F; z_1) < P(G; z_1)$ when the alternative value of $z_1 \in Z$ is chosen. However, it may be the case that the ranking which holds at some poverty line $z \in Z$ is not reversed at any other feasible poverty standard. The poverty verdict would then not be in doubt.

To develop this argument we introduce the *poverty ordering* $\boldsymbol{P(Z)}$ defined by[5]

$$(5) \qquad F\boldsymbol{P(Z)}G \quad \text{if and only if} \quad P(F; z) \leqslant P(G; z) \quad \text{for all } z \in Z$$
$$\text{and} \quad P(F; z) < P(G; z) \quad \text{for some } z \in Z,$$

where $F\boldsymbol{P(Z)}G$ means that F has unambiguously less poverty than G with respect to the poverty index P and the range Z. It is natural to begin by considering the case $Z = \mathbb{R}_{++}$, so that a poverty comparison is inconclusive if the ranking obtained at one poverty line is reversed at any other positive poverty standard. This represents an extremely cautious viewpoint, which refrains from making a judgement if there is any suspicion that the verdict may be challenged. Yet unambiguous poverty judgements can still be made, and the corresponding poverty ordering $\boldsymbol{P} := \boldsymbol{P}(\mathbb{R}_{++})$ may have significant power to rank distributions, as we demonstrate in the context of the indices P_α.

4. THE POVERTY ORDERING P_α

Given $F \in \mathscr{F}$, set $F_1 := F$ and let F_α be defined recursively for $\alpha \geqslant 2$ by $F_\alpha(s) := \int_0^s F_{\alpha-1}(t)\,dt$. Following Fishburn (1980), define the α-degree stochastic dominance relation $\boldsymbol{D_\alpha}$ for positive integer α by

$$(6) \qquad F\boldsymbol{D_\alpha}G \quad \text{if and only if} \quad F_\alpha(s) \leqslant G_\alpha(s) \quad \text{for all } s > 0$$
$$\text{and} \quad F_\alpha(s) < G_\alpha(s) \quad \text{for some } s > 0.$$

By a change of variable, we see that $z^{\alpha-1}P_\alpha(F; z) = \int_0^z (z-y)^{\alpha-1}\,dF(y)$. Furthermore, by repeatedly integrating by parts, it can be shown that

$$\int_0^z (z-y)^{\alpha-1} dF(y) = (\alpha-1)!F_\alpha(z).$$[6]

A comparison of (5) and (6) then yields the following result.

PROPOSITION 1: *For any positive integer* α: $F\boldsymbol{P_\alpha}G$ *iff* $F\boldsymbol{D_\alpha}G$.

Thus the poverty ordering $\boldsymbol{P_\alpha}$ is precisely the stochastic dominance ordering $\boldsymbol{D_\alpha}$.

One immediate implication of this proposition is that the poverty orderings of different indices may be closely related. The orderings $\boldsymbol{P_\alpha}$ are nested, in that $F\boldsymbol{P_\alpha}G$ implies $F\boldsymbol{P_\beta}G$ whenever α and β are positive integers satisfying $\alpha \leqslant \beta$. Thus, for example, if one income distribution has unambiguously less poverty according to the headcount ratio P_1, then the same must be true for the per capita income gap index P_2 as well as for the distribution

[5] Note that $\boldsymbol{P(Z)}$ is a transitive, asymmetric, but not necessarily complete relation, and hence is a strict partial ordering in the terminology of Sen (1970). For the sake of brevity, we often substitute "ordering" for "strict partial ordering."

[6] This expression is quite standard in the stochastic dominance literature: see Fishburn (1976) or O'Brien (1984). Note that the change in variable is valid even if the distribution function F is discontinuous.

sensitive index P_3. In this sense the partial orderings $\boldsymbol{P}_\alpha$ become more complete, and potentially more applicable, as α grows larger.

The relationship between the poverty orderings $\boldsymbol{P}_\alpha$ and stochastic dominance also enables us to obtain interesting interpretations of $\boldsymbol{P}_\alpha$ in terms of social welfare. Suppose $\mathscr{U}$ denotes the class of all welfare functions of the form $U(F) = \int u(x)\,dF(x)$, where $u: \mathbb{R}_+ \to \mathbb{R}$ is any continuous function. Let $\mathscr{U}_1 \subset \mathscr{U}$ be defined by the condition $u' > 0$; let $\mathscr{U}_2 \subset \mathscr{U}_1$ be defined by $u'' < 0$; and let $\mathscr{U}_3 \subset \mathscr{U}_2$ be defined by $u''' > 0$. Finally, for $\alpha = 1$, 2, and 3, let $\boldsymbol{U}_\alpha$ be the partial ordering

$$F\,\boldsymbol{U}_\alpha\,G \quad \text{if and only if} \quad U(F) > U(G) \quad \text{for all } U \in \mathscr{U}_\alpha .$$

Applying well-known results on stochastic dominance (e.g., Bawa (1975)), we obtain the following proposition:

Proposition 2: *For* $\alpha = 1, 2,$ *and* 3: $F\boldsymbol{P}_\alpha G$ *iff* $F\boldsymbol{U}_\alpha G$.

Thus the observation that F has unambiguously less poverty than G according to P_α is equivalent to the statement that F is better than G for all welfare functions in $\mathscr{U}_\alpha$.

The set $\mathscr{U}$ is often interpreted as the class of (symmetric) utilitarian welfare functions, while $u' > 0$ is a monotonicity condition which indicates approval of any rise in individual incomes. In this respect, $\boldsymbol{U}_1$ can be regarded as the *welfare ordering* that corresponds to unanimous agreement among all monotonic utilitarian functions. Similarly, the condition $u'' < 0$ is equivalent to the "equality-preference" requirement that any mean-preserving progressive transfer raises social welfare. Proposition 2 therefore establishes that $F\boldsymbol{P}_2 G$ if and only if F is judged better than G by all utilitarian welfare functions that are monotonic and equality-preferring. Finally, the condition $u''' > 0$ corresponds to the assumption that social welfare is "transfer sensitive" and hence gives more weight to transfers that occur at lower income levels. (See, for example, Kolm (1976) or Shorrocks and Foster (1987)). The poverty ordering based on the "distribution sensitive" index P_3 is therefore equivalent to the welfare ordering based on the set of utilitarian functions satisfying monotonicity, equality-preference, and transfer sensitivity.

The poverty ordering $\boldsymbol{P}_2$ can also be linked to two other important constructs used in the analysis of income distribution, the Lorenz curve and the generalized Lorenz curve. The generalized Lorenz curve of a distribution F is defined by

$$(7) \qquad GL(F; p) := \int_0^p F^{-1}(q)\,dq, \quad \text{for } p \in [0,1],$$

and its associated partial ordering $\boldsymbol{GL}$ is given as:

$$(8) \qquad F\,\boldsymbol{GL}\,G \quad \text{if and only if} \quad GL(F;p) \geq GL(G;p) \quad \text{for all } p \in [0,1],$$

$$\text{and} \quad GL(F;p) > GL(G;p) \quad \text{for some } p \in [0,1].$$

In other words, $F\boldsymbol{GL}\,G$ holds if the mean income of the poorest $100p$ per cent of the population in F is no smaller than that in G, and for some p it is larger. Since it can be shown that $\int_0^z [G(t) - F(t)]\,dt$ is nonnegative for all z if and only if $\int_0^p [F^{-1}(q) - G^{-1}(q)]\,dq$ is nonnegative for all p, it follows that $F\boldsymbol{P}_2 G$ iff $F\boldsymbol{GL}\,G$. Thus the unambiguous poverty ordering associated with the per capita income gap index is simply generalized Lorenz dominance.

The special case $\mu_F = \mu_G$ is worthy of special attention, since in this circumstance the generalized Lorenz ordering $\boldsymbol{GL}$ degenerates into the more familiar Lorenz ordering popularized by Atkinson (1970). For distributions with a common mean, therefore, $F\boldsymbol{P}_2 G$

(and hence $F U_2 G$) holds if and only if F is unambiguously more equal than G. Thus if we follow Atkinson in adopting the class of welfare functions $\mathscr{U}_2$, we discover that an unambiguous welfare ranking implies not only an unambiguous inequality ranking, but also an unambiguous poverty ranking based on the per capita income gap index P_2.

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