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ETHICAL INDICES FOR THE MEASUREMENT OF POVERTY

BY CHARLES BLACKORBY AND DAVID DONALDSON[1]

This paper generalizes the poverty index introduced by Sen and demonstrates that (i) for every homothetic social evaluation function there is one relative poverty index, (ii) Sen's index is a relative poverty index and corresponds to a Gini social evaluation function, (iii) for every translatable social evaluation function there is one absolute poverty index, and (iv) ethical content in these poverty indices requires that the social evaluation function be structured so that any group of poor people is strictly separable from anyone richer.

IN A RECENT PAPER, Sen [**10**] has suggested an index of poverty based on an ordinal approach to welfare comparisons. Given a poverty line, a priori, this index has several appealing properties: (i) it can be computed using readily available information, (ii) it is sensitive to the percentage of the population that is below the line (the "head-count ratio"), (iii) it depends on the income of the average poor person, and (iv) it depends on the amount of inequality among the poor themselves.

In this note, we offer an alternative interpretation and a generalization of Sen's index as an "ethical index." These are indices, usually of inequality, that are exact for social evaluation functions. Each index is thus implied by and implies at least one social evaluation function.

Essential to the construction of these ethical indices is the notion of the *representative income*[2] of the poor.

This representative income is an inequality-adjusted per-capita income; that is, it is that level of income which, if given to each poor person, would prove ethically equivalent to the current distribution. There is a one-to-one relationship between the representative income of a group and its social evaluation function (provided, of course, that there is a function for each group).

In Section 1 we show that there is a Sen-like poverty index for every social evaluation function defined over the poor. In Section 2 we discuss the relationship between social evaluation functions defined over the poor to those defined over all of society. We conclude that the social evaluation function must be completely strictly recursive in the sense that the ordering over any group of poor people must be separable from (the income of) anyone who is richer. In Section 3, we note that the above class of indices consists of "relative" poverty indices, analogous to indices of relative inequality. We then define "absolute" indices of poverty and the class of social evaluation functions which generates them. We also present several absolute poverty indices.

Before proceeding to our argument, we summarize the relationship between indices of relative inequality, homothetic social evaluation functions, and representative incomes.[3] Let y^p be the income vector of the poor, and let W^p be

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[2] For the population as a whole, this representative income is the same as the Atkinson–Kolm–Sen "equally distributed equivalent income." See Atkinson [**1**], Kolm [**6**], and Sen [**9**].

[3] This is discussed in detail in Blackorby and Donaldson [**2**].

their homothetic (ordinal) social evaluation function.[4] Hence,

$$W^p(y^p) = \phi(\bar{W}^p(y^p)) \tag{1}$$

where ϕ is increasing and $\bar{W}^p$ is positively linearly homogeneous. The representative income of the poor is that income, ξ^p, which, if enjoyed by each poor person, is ranked as socially indifferent to the current distribution. Hence, ξ^p is defined by

$$W^p(\xi^p \underline{1}) = W^p(y^p)$$

or

$$\bar{W}^p(\xi^p \underline{1}) = \bar{W}^p(y^p), \tag{2}$$

since ϕ is increasing. $\underline{1}$ is an appropriate vector of ones. Since $\bar{W}$ is positively linearly homogeneous,

$$\xi^p = \frac{\bar{W}^p(y^p)}{\bar{W}^p(\underline{1})} = \Xi^p(y^p). \tag{3}$$

Hence, given a functional form for W^p or Ξ^p we can compute the other using (1) and (3).[5]

The index of relative inequality among the poor corresponding to W^p is

$$I^p(y^p) = \frac{\mu^p - \xi^p}{\mu^p} = \frac{\mu^p - \Xi^p(y^p)}{\mu^p},^{6} \tag{4}$$

where μ^p is the mean of y^p. I^p is homogeneous of degree zero (i.e., it is a relative index) because Ξ^p is homogeneous of degree one. Further, I^p is S-convex (it agrees with the Lorenz quasi-ordering) if W^p is S-concave and, in this case, it is symmetric and ranges between zero and one with a value of zero at equality. Given a functional form for I^p, we can find Ξ^p and W^p from (4), (3), and (1).

1. RELATIVE POVERTY INDICES

We define a relative poverty index as one whose value is unchanged when all incomes and the poverty line itself are multiplied by a positive scalar. First we analyze Sen's poverty index. Suppose that the poverty line is a pre-specified income level z, that the total population has N members, that the set of the poor

[4] Homotheticity is not necessary for what follows but does make the argument quite simple. For an analysis of the non-homothetic case, see Blackorby and Donaldson [**2**, **3**].

[5] Since W is an ordinal function Ξ^p and W^p are ordinally equivalent.

[6] W^p is continuous, S-concave, homothetic, and increasing along rays if and only if I^p is continuous, S-convex, homogeneous of degree zero, and "reference-level free"; Blackorby and Donaldson [**2**, Propositions 1, 2, 3, and 4].

(those people whose incomes are at or below the poverty line) is $Z(z)$, and that $Z(z)$ has $n(z)$ members. Let W_g^p, Ξ_g^p, and I_g^p be the *Gini* social evaluation function, representative income, and index of inequality for the poor (see Blackorby and Donaldson [**2**, Section III]).

The index suggested by Sen [**10**, p. 223] is defined by

$$(5)\qquad P_g(y^p)=\frac{n(z)}{N}\left[\sum_{i\in Z(z)}\left(\frac{z-y_i}{n(z)z}\right)+\left(1-\sum_{i\in Z(z)}\left(\frac{z-y_i}{n(z)z}\right)\right)I_g^p(y^p)\right]$$
$$=\frac{n(z)}{N}\left[\frac{z-\xi_g^p}{z}\right],^{7}$$

where

$$(6)\qquad \xi_g^p=\Xi_g^p(y^p)=\frac{1}{[n(z)]^2}\sum_{i\in Z(z)}(2i-1)\tilde{y}_i$$

and $\tilde{y}$ is a permutation of the elements of y so that

$$\tilde{y}_1\geqslant\tilde{y}_2\geqslant\ldots\ldots\geqslant\tilde{y}_{n(z)}.$$

ξ^p is the representative income of the poor as measured by the Gini social evaluation function defined over the poor alone. Note that $\xi_g^p\leqslant\mu^p$ with equality only if the poor all have the same income (μ^p). P_g is thus the percentage shortfall of the representative income of the poor from the poverty line multiplied by the "head-count ratio."

Using definitions (2)–(4), a general relative poverty index may be defined as

$$(7)\qquad P(y^p)=f\left[\frac{n(z)}{N},\frac{z-\xi^p}{z}\right]$$

where ξ^p is the representative income of the poor as measured by an arbitrary (homothetic) social evaluation function. P is homogeneous of degree zero in y^p and z because $\xi^p=\Xi^p(y^p)$ is homogeneous of degree one. We will require that f be homogeneous of degree one in $n(z)/N$ (doubling the head-count ratio ceteris paribus doubles the index) *and* in $(z-\xi^p)/z$ (doubling the percentage shortfall ceteris paribus doubles the index) and that $f[1,1]=1$. In that case we have

$$(8)\qquad P(y^p)=\frac{n(z)}{N}\left[\frac{z-\xi^p}{z}\right].$$

[7] Note that

$$\sum_{i\in Z(z)}\left(\frac{z-y_i}{n(z)z}\right)=1-\mu_p/z$$

where μ_p is the mean of the incomes of the poor. Hence

$$P_g(y^p)=\frac{n(z)}{N}\left[1-\frac{\mu_p}{z}+\frac{\mu_p I_g^p(y^p)}{z}\right]$$

which yields (5) using (4).

For example, if the social evaluation function is the symmetric mean of order r, then

$$\xi_r^p = \Xi_r^p(y^p) = \begin{cases} \left[\sum_{i \in Z(z)} \frac{1}{n(z)} y_i^r \right]^{1/r}, & r \leqslant 1, r \neq 0, \\ \prod_{i \in Z(z)} y_i^{1/n(z)}, & r = 0. \end{cases} \tag{9}$$

Given (8), it is clear that to every homothetic social evaluation function there corresponds a different relative poverty index. These indices differ only in the way in which the amount of relative inequality among the poor is accounted for. This general index retains all of the nice properties of Sen's index: (i) it is sensitive to the head-count ratio, (ii) it is sensitive to how poor the poor are, and (iii) it is sensitive to the amount of inequality among the poor themselves. This is true, of course, because ξ^p is μ^p multiplied by I^p. In addition, given a functional form for W, like the mean of order r, this index is easy to compute. Further, if the social evaluation function for the poor has ethical content (see the next section), so also does the poverty index.

2. THE ETHICAL INTERPRETATION OF POVERTY INDICES

In Section 1, we found a relative poverty index for every homothetic social evaluation function defined over the poor. In order that that function be ethically significant it must reflect a set of ethical judgements for the whole society. Suppose that there exists a social evaluation function $W: \Omega^N \to R$ (where Ω^N is the non-negative Euclidian N-orthant). In order that the social evaluation function for the poor depend only on the incomes of the poor and not on who they are, W must be symmetric. Thus we may write $\tilde{y}$ as the welfare-ranked permutation of y $(\tilde{y}_1 \geqslant \tilde{y}_2 \geqslant \ldots \geqslant \tilde{y}_n)$ and we know that $W(\tilde{y}) = W(y)$. If $\tilde{y}^p$ is the welfare-ranked vector of the incomes of the poor and $\tilde{y}^R$ is the welfare-ranked vector of the non-poor, then $\tilde{y} = (\tilde{y}^R, \tilde{y}^p)$. By fixing $\tilde{y}^R$ at $\bar{\tilde{y}}^R$, we automatically generate from W a social evaluation function for the poor $\hat{W}^p$ and its image is

$$\hat{W}^p(y^p) \equiv W(\bar{\tilde{y}}^R, \tilde{y}^p). \tag{10}$$

In this case, the social evaluation function for the poor depends not only upon the incomes of those below the poverty line, but also upon the incomes of the non-poor. Hence, if any element of $\bar{\tilde{y}}^R$ were to change, the social evaluation function for the poor would change. If W is continuous and nondecreasing, then the minimal condition needed for $\hat{W}^p$ not to depend upon the income of the non-poor is that the poor be separable from the non-poor. Then the social evaluation function can be written as

$$W(\tilde{y}) = \overset{*}{W}(\tilde{y}^R, W^p(\tilde{y}^p)). \tag{11}$$

This, however, is not enough to guarantee that $\overset{*}{W}$ be increasing in $W^p(\tilde{y}^p)$. This seems to be an important requirement; surely there is little point in being concerned with poverty indices if the general social evaluation is insensitive to the plight of the poor. To ensure this result it is necessary to assume that the poor be strictly separable from the non-poor (for a discussion of these notions, see Blackorby, Primont, and Russell [**4**; **5**, Chapter 3]). In this latter case it is not necessary to maintain any regularity conditions on W, and W can be written as in (11) where $\overset{*}{W}$ is increasing in $W^p(\tilde{y}^p)$. Note, however, that $\overset{*}{W}$ and W^p both depend on the poverty line itself. To avoid this we assume that every group of poor people is strictly separable from anyone with a higher income. This implies that the social evaluation function (when restricted to a welfare-ranked subset of Ω^N) is completely strictly recursive (see Blackorby, Primont, and Russell [**5**, Chapter 6]), in the discrete partition; that is,

$$W(\tilde{y}) = W^1(\tilde{y}_1, w^2), \tag{12a}$$

where

$$w^r = W^r(\tilde{y}_r, w^{r+1}) \qquad (r = 2, \ldots, N-1), \tag{12b}$$

$$w^N = W^N(\tilde{y}^N), \tag{12c}$$

and W^r is increasing in w^{r+1}. In addition, if W is homothetic (homogeneous) then each W^r, $r = 1, \ldots, N$, may be chosen to be homothetic (homogeneous) as well. Further, S-concavity of W^p for every group of poor people is guaranteed by S-concavity of W.

This leaves quite a rich class of functions from which to choose; not only are all completely separable (additive) functions in the class, but so also are some non-separable functions like the generalized Gini[8] (of which the Gini is a special case).

The generalized Gini can be written as

$$W_G(y) = W_G(\tilde{y}) = \alpha_1\tilde{y}_1 + \alpha_2\tilde{y}_2 + \ldots + \alpha_N\tilde{y}_N \tag{13}$$

where $\alpha_i \geqslant 0$ for all i. If $\alpha_i = (2_i - 1)/N^2$, then $W_g = W_G$. If $W = W_G$, then the social evaluation function for the poor is also a generalized Gini. If $W = W_g$, then the social evaluation function for the poor is a generalized Gini but not the Gini itself unless the poor comprise the whole of society.

3. ABSOLUTE POVERTY INDICES

Sen's poverty index and the generalization which we have suggested are relative indices of poverty. They are homogeneous of degree zero in the incomes of the poor and the poverty line. To do this, they depend on the percentage shortfall of the representative income of the poor from the poverty line and the head-count ratio of the poor. For many policy purposes it might be convenient to have an index based on the absolute shortfall of the representative income of the poor from the poverty line and the absolute number of poor people. We define an

[8] The generalized Gini was pointed out to us by John Waymark [**11**].

absolute index to be

$$(14) \qquad n(z)[z-\xi^p].$$

If the social evaluation function is homothetic, this index is homogeneous of degree one in y^p and z.

It is possible, however, to construct an absolute index with a property that makes it depend on the level of absolute inequality among the poor. That is, we make it invariant to any change in y^p that preserves the absolute (as opposed to the relative) differentials among the poor. Thus $(y^p+\lambda\underline{1})$ and y^p have the same absolute inequality, where λ is an arbitrary scalar. To do this, we assume that the social evaluation function for the poor is translatable and defined on all of R^n.[9] W^p is translatable if it can be written as

$$(15) \qquad W^p(y^p)=\phi(\tilde{W}^p(y^p))$$

where ϕ is increasing in its argument and

$$(16) \qquad \tilde{W}^p(y^p+\lambda\underline{1})=\tilde{W}^p(y^p)+\lambda$$

for all real λ. $\tilde{W}^p$ is said to be unit-translatable.

The representative income of the poor is defined by

$$W^p(\xi^p\underline{1})=W^p(y^p)$$

or

$$(17) \qquad \tilde{W}^p(\xi^p\underline{1})=\tilde{W}^p(y^p).$$

Since $\tilde{W}^p$ is unit-translatable,

$$(18) \qquad \xi^p=\Lambda^p(y^p)=\tilde{W}^p(y^p)-\tilde{W}^p(\underline{0})$$

where $\underline{0}$ is an appropriate vector of zeros, Λ^p is unit-translatable.

The absolute index of inequality among the poor is

$$(19) \qquad A^p(y^p)=\mu^p-\xi^p=\mu^p-\Lambda^p(y^p).$$ [10]

A^p is translation invariant because Λ^p is unit-translatable. Further, A^p is S-convex (it agrees with the Lorenz quasi-ordering) if W^p is S-concave and, in this case, it is symmetric, non-negative, and equal to zero at equality. Given a functional form for A^p we can find Λ^p and W^p from (17), (18), and (19).

The absolute poverty index, $Q(y^p)$, is defined by

$$(20) \qquad \begin{aligned} Q(y^p)&=n(z)[z-\xi^p]\\ &=n(z)[z-\mu^p+A^p(y^p)]. \end{aligned}$$

[9] Absolute indices can be constructed from non-translatable functions defined on Ω^n only. For a careful discussion, see Blackorby and Donaldson [**3**]. The idea of absolute indices is apparently due to Kolm and is discussed in Kolm [**7**, **8**].

[10] W^p is continuous, S-concave, translatable, and increasing along translating rays, if and only if A^p is continuous, S-convex, translation invariant, and "reference-level free"; Blackorby and Donaldson [**3**, Theorems 3, 4, 5, 6, and 7].

Q is invariant with respect to translation of z and y^p (it depends only on absolute differentials), and $A^p(y^p)$ is an addition to the absolute shortfall of μ^p from the poverty line to adjust the per capita income of the poor for inequality among the poor. $Q(y^p)$ is easily interpreted as the dollar cost of poverty. If each poor person were given $(z-\xi^p)$ dollars, then the index would be zero (because Λ^p is unit-translatable) at an aggregate cost of $Q(y^p)$. Hence an absolute poverty index that depends on absolute differentials only among the poor exists for every translatable social evaluation function defined on the incomes of the poor.

If these indices are to have ethical content, then they must (again) be consistent with a set of ethical judgements for the whole society. The same considerations of Section 2 apply to these indices as well, and, by the same arguments, we will require that W be symmetric, completely strictly recursive in any welfare-ranked subset of R^N, and S-concave.

An example of an absolute index is the absolute Gini, since the Gini social evaluation function is translatable as well as homothetic. The absolute Gini poverty index is

$$Q_g(y^p)=n(z)[z-\xi_g^p] \tag{21}$$

where ξ_g^p is defined by (6). This index can have ethical content but it cannot come from a Gini social evaluation function for all of society. Therefore, a generalized Gini seems more appropriate. Another alternative arises from the Kolm-Pollak social evaluation function.[11] Its implicit absolute poverty index is

$$Q_{kp}(y^p)=n(z)[z-\xi_{kp}^p] \tag{22}$$

where

$$\xi_{kp}^p=\Lambda_{kp}^p(y^p)=-\frac{1}{\gamma}\ln\left[\sum_{i\in Z(z)}\frac{e^{-\gamma y_i}}{n(z)}\right], \tag{23}$$

$\gamma>0$. γ is a free parameter which determines the curvature of the social indifference surfaces. Λ_{kp}^p is completely (additively) separable, and hence, completely strictly recursive in any welfare-ranked subset of R^N as required above.

4. CONCLUSION

We have generalized Sen's ordinal poverty index in several ways in this paper and have shown that: (i) for every homothetic social evaluation function there is one relative poverty index, (ii) Sen's index is the relative poverty index corresponding to the Gini social evaluation function for the poor, (iii) for every translatable social evaluation function there is a single absolute index of poverty that is invariant with respect to translation of z and y^p, (iv) ethical content in the poverty indices requires that the overall social evaluation function be structured so that any group of poor people is strictly separable from anyone richer.

[11] This is discussed in Blackorby and Donaldson [3].

It should be noted that Sen's poverty index was intended as an ordinal measure of personal welfare and suffering without any essential use of the concept of social welfare. The generalizations which we have suggested are also amenable to this interpretation.

One minor problem remains. The population may not consist only of individuals facing a single poverty line. There may be different poverty lines for families of different compositions. A solution to this problem is to convert the population into a population of adult equivalents. For example, let $z(k, l)$ be the poverty line for a family of k adults and l children. If z is the poverty line for a single adult, define $z(k, l)/z$ as the number of adult equivalents in the family. Divide, too, the family income by this number and assign this to each member of the family. This procedure allows the above formulae to be used in the construction of poverty indices.

University of British Columbia

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